

GEOMETRIC REALIZATIONS OF LUSZTIG'S SYMMETRIES

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ABSTRACT. In this paper, we give geometric realizations of Lusztig's symmetries. We also give projective resolutions of a kind of standard modules. By using the geometric realizations and the projective resolutions, we obtain the categorification of the formulas of Lusztig's symmetries.

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1. INTRODUCTION

Let \mathbf{U} be the quantum group and \mathbf{f} be the Lusztig's algebra associated with a Cartan datum. Denote by \mathbf{U}^+ and \mathbf{U}^- the positive part and the negative part of \mathbf{U} respectively. There are two well-defined $\mathbb{Q}(v)$ -algebra homomorphisms $^+ : \mathbf{f} \rightarrow \mathbf{U}$ and $^- : \mathbf{f} \rightarrow \mathbf{U}$ with images \mathbf{U}^+ and \mathbf{U}^- respectively.

Lusztig introduced the canonical basis \mathbf{B} of \mathbf{f} in [11, 13, 16]. Let $Q = (I, H)$ be a quiver corresponding to \mathbf{f} and \mathbf{V} be an I -graded vector space such that $\dim \mathbf{V} = \nu \in \mathbb{N}I$. He studied the variety $E_{\mathbf{V}}$ consisting of representations of Q with dimension vector ν , and a category $\mathcal{Q}_{\mathbf{V}}$ of some semisimple perverse sheaves on $E_{\mathbf{V}}$. Let $K(\mathcal{Q}_{\mathbf{V}})$ be the Grothendieck group of $\mathcal{Q}_{\mathbf{V}}$. Considering all dimension vectors, he proved that $\bigoplus_{\nu \in \mathbb{N}I} K(\mathcal{Q}_{\mathbf{V}})$ realizes \mathbf{f} and the set of isomorphism classes of simple objects realizes the canonical basis \mathbf{B} .

Lusztig also introduced some symmetries T_i on \mathbf{U} for all $i \in I$ in [10, 12]. Note that $T_i(\mathbf{U}^+)$ is not contained in \mathbf{U}^+ . Hence, Lusztig introduced two subalgebras ${}_i\mathbf{f}$ and ${}^i\mathbf{f}$ of \mathbf{f} for any $i \in I$, where ${}_i\mathbf{f} = \{x \in \mathbf{f} \mid T_i(x^+) \in \mathbf{U}^+\}$ and ${}^i\mathbf{f} = \{x \in \mathbf{f} \mid T_i^{-1}(x^+) \in \mathbf{U}^+\}$. Let $T_i : {}_i\mathbf{f} \rightarrow {}^i\mathbf{f}$ be the unique map satisfying $T_i(x^+) = T_i(x)^+$. The algebra \mathbf{f} has the following direct sum decompositions $\mathbf{f} = {}_i\mathbf{f} \oplus \theta_i\mathbf{f} = {}^i\mathbf{f} \oplus \mathbf{f}\theta_i$. Denote by ${}_i\pi : \mathbf{f} \rightarrow {}_i\mathbf{f}$ and ${}^i\pi : \mathbf{f} \rightarrow {}^i\mathbf{f}$ the natural projections.

Associated to a finite dimensional hereditary algebra Λ , Ringel introduced the Hall algebra and the composition subalgebra \mathcal{F} in [18], which gives a realization of the positive part of the quantum group \mathbf{U} . If we use the notations of Lusztig in [15], we have the canonical isomorphism between the composition subalgebra \mathcal{F} and the Lusztig's algebra \mathbf{f} . Via the Hall algebra approach, one can apply BGP-reflection functors to quantum groups to give precise constructions of Lusztig's symmetries ([19, 15, 22, 24, 3, 25]).

To a Lusztig's algebra \mathbf{f} , Khovanov, Lauda ([6]) and Rouquier ([20]) introduced a series of algebras \mathbf{R}_{ν} respectively. The category of finitely generated projective modules of \mathbf{R}_{ν} gives a categorification of \mathbf{f} and \mathbf{R}_{ν} are called Khovanov-Lauda-Rouquier (KLR) algebras. Varagnolo, Vasserot ([23]) and Rouquier ([21]) realized the KLR algebra \mathbf{R}_{ν} as the extension algebra of semisimple perverse sheaves in $\mathcal{Q}_{\mathbf{V}}$ and proved that the set of indecomposable projective modules of \mathbf{R}_{ν} can categorify the canonical basis \mathbf{B} .

In [4, 5], Kato gave the categorification of the PBW-type bases of quantum groups of finite type. He constructed some modules (which are called standard modules) of the KLR algebras \mathbf{R}_{ν} and proved that there standard modules can categorify the PBW-type basis of \mathbf{f} by using the geometric realizations of \mathbf{R}_{ν} given by Varagnolo, Vasserot and Rouquier. He proved that the length of the projective resolution of any standard module is finite, which is the categorification of the following fact: the transition matrix between the PBW-type basis of \mathbf{f} and the canonical basis \mathbf{B} is triangular with diagonal entries equal to 1. This result implies that the global dimensions of the KLR algebras \mathbf{R}_{ν} are also finite. In [17, 2], Brundan, Kleshchev and McNamara proved the same result by using an algebraic method.

Let $i \in I$ be a sink (resp. source) of Q . Similarly to the geometric realization of \mathbf{f} , consider a subvariety ${}^iE_{\mathbf{V}}$ (resp. ${}^iE_{\mathbf{V}}$) of $E_{\mathbf{V}}$ and a category ${}^i\mathcal{Q}_{\mathbf{V}}$ (resp. ${}^i\mathcal{Q}_{\mathbf{V}}$) of some semisimple perverse sheaves on ${}^iE_{\mathbf{V}}$ (resp. ${}^iE_{\mathbf{V}}$). In Section 3.2, we verify that $\bigoplus_{\nu \in \mathbb{N}^I} K({}^i\mathcal{Q}_{\mathbf{V}})$ (resp. $\bigoplus_{\nu \in \mathbb{N}^I} K({}^i\mathcal{Q}_{\mathbf{V}})$) realizes ${}^i\mathbf{f}$ (resp. ${}^i\mathbf{f}$).

Let $i \in I$ be a sink of Q . Let $Q' = \sigma_i Q$ be the quiver by reversing the directions of all arrows in Q containing i . Hence, i is a source of Q' . Consider two I -graded vector spaces \mathbf{V} and \mathbf{V}' such that $\dim \mathbf{V}' = s_i(\dim \mathbf{V})$. In the case of finite type, Kato introduced an equivalence $\tilde{\omega}_i : {}^i\mathcal{Q}_{\mathbf{V},Q} \rightarrow {}^i\mathcal{Q}_{\mathbf{V}',Q'}$ and studied the properties of this equivalence in [4, 5]. In this paper, we generalize his construction to all cases and prove that the map induced by $\tilde{\omega}_i$ realizes the Lusztig's symmetry $T_i : {}^i\mathbf{f} \rightarrow {}^i\mathbf{f}$. For the proof of the result, we shall study the relations between the map induced by $\tilde{\omega}_i$ and the Hall algebra approach to T_i in [15].

In [14], Lusztig showed that Lusztig's symmetries and canonical bases are compatible. Let ${}^i\mathbf{B} = {}^i\pi(\mathbf{B})$, which is a $\mathbb{Q}(v)$ -basis of ${}^i\mathbf{f}$. Similarly, ${}^i\mathbf{B} = {}^i\pi(\mathbf{B})$ is a $\mathbb{Q}(v)$ -basis of ${}^i\mathbf{f}$. Lusztig proved that $T_i : {}^i\mathbf{f} \rightarrow {}^i\mathbf{f}$ maps any element of ${}^i\mathbf{B}$ to an element of ${}^i\mathbf{B}$.

For any simple perverse sheaf \mathcal{L} in $\mathcal{Q}_{\mathbf{V},Q}$, the restriction ${}^i\mathcal{L} = j_{\mathbf{V}}^*(\mathcal{L})$ on ${}^iE_{\mathbf{V},Q}$ is also a simple perverse sheaf and belongs to ${}^i\mathcal{Q}_{\mathbf{V},Q}$, where $j_{\mathbf{V}} : {}^iE_{\mathbf{V},Q} \rightarrow E_{\mathbf{V},Q}$ is the canonical embedding. Let ${}^i\mathcal{L} = \tilde{\omega}_i({}^i\mathcal{L}) \in {}^i\mathcal{Q}_{\mathbf{V}',Q'}$. The simple perverse sheaf ${}^i\mathcal{L}$ can be wrote as ${}^i\mathcal{L} = j_{\mathbf{V}'}^*(\mathcal{L}')$, where \mathcal{L}' is a simple perverse sheaf in $\mathcal{Q}_{\mathbf{V}',Q'}$ and $j_{\mathbf{V}'} : {}^iE_{\mathbf{V}',Q'} \rightarrow E_{\mathbf{V}',Q'}$ is the canonical embedding. Since the map induced by $\tilde{\omega}_i$ realizes $T_i : {}^i\mathbf{f} \rightarrow {}^i\mathbf{f}$, this result gives a geometric interpretation of Lusztig's result in [14].

For any $m \leq -a_{ij}$, let

$$f(i, j; m) = \sum_{r+s=m} (-1)^r v^{-r(-a_{ij}-m+1)} \theta_i^{(r)} \theta_j \theta_i^{(s)} \in \mathbf{f},$$

and

$$f'(i, j; m) = \sum_{r+s=m} (-1)^r v^{-r(-a_{ij}-m+1)} \theta_i^{(s)} \theta_j \theta_i^{(r)} \in \mathbf{f}.$$

In [16], Lusztig proved that $T_i(f(i, j; m)) = f'(i, j; m')$, where $m' = -a_{ij} - m$. The following formula

$$T_i(E_j) = \sum_{r+s=-a_{ij}} (-1)^r v^{-r} E_i^{(s)} E_j E_i^{(r)}$$

is a special case of $T_i(f(i, j; m)) = f'(i, j; m')$. In this paper, our main result is the categorification of these formulas. Consider an I -graded vector space \mathbf{V} such that $\dim \mathbf{V} = \nu = mi + j$. Let $\mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})$ be the bounded $G_{\mathbf{V}}$ -equivariant derived category of complexes of l -adic sheaves on $E_{\mathbf{V}}$. We construct a series of distinguished triangles in $\mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})$, which represent the constant sheaf $\mathbf{1}_{E_{\mathbf{V}}}$ in terms of some semisimple perverse sheaves $I_p \in \mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})$ geometrically. Note that, $\mathbf{1}_{E_{\mathbf{V}}}$ corresponds to a standard module K_m of the KLR algebra \mathbf{R}_{ν} and I_p correspond to projective modules of \mathbf{R}_{ν} . This result means that we find projective resolutions of the standard modules

K_m . Consider two I -graded vector spaces \mathbf{V} and \mathbf{V}' such that $\underline{\dim} \mathbf{V} = mi + j$ and $\underline{\dim} \mathbf{V}' = s_i(\underline{\dim} \mathbf{V}) = m'i + j$. Applying to the Grothendieck group, $\mathbf{1}_{E_{\mathbf{V}, Q}}$ (resp. $\mathbf{1}_{E_{\mathbf{V}', Q'}}$) corresponds to $f(i, j; m)$ (resp. $f'(i, j; m')$). The property of BGP-reflection functors implies $\tilde{\omega}_i(v^{-mN} \mathbf{1}_{E_{\mathbf{V}, Q}}) = v^{-m'N} \mathbf{1}_{E_{\mathbf{V}', Q'}}$, therefore $T_i(f(i, j; m)) = f'(i, j; m')$.

In Example D of [5], Kato constructed a short exact sequence

$$0 \longrightarrow P_1 * P_2[2] \longrightarrow P_2 * P_1 \longrightarrow Q_{12} \longrightarrow 0$$

which coincides with the projection resolution in our main result in the case of finite type. In Theorem 4.10 of [2], Brundan, Kleshchev and McNamara constructed a shout exact sequence of standard modules

$$0 \longrightarrow v^{-\beta \cdot \gamma} \Delta(\beta) \circ \Delta(\gamma) \longrightarrow \Delta(\gamma) \circ \Delta(\beta) \longrightarrow [p_{\beta, \gamma} + 1] \Delta(\alpha) \longrightarrow 0.$$

In the case of finite type, the projection resolution in our main result is a special case of the shout exact sequence above where $\alpha = \alpha_i + \alpha_j$.

2. QUANTUM GROUPS AND LUSZTIG'S SYMMETRIES

2.1. Quantum groups. Let I be a finite index set with $|I| = n$ and $A = (a_{ij})_{i, j \in I}$ be a generalized Cartan matrix. Let $(A, \Pi, \Pi^\vee, P, P^\vee)$ be a Cartan datum associated with A , where

- (1) $\Pi = \{\alpha_i \mid i \in I\}$ is the set of simple roots;
- (2) $\Pi^\vee = \{h_i \mid i \in I\}$ is the set of simple coroots;
- (3) P is the weight lattice;
- (4) P^\vee is the dual weight lattice.

In this paper, we always assume that the generalized Cartan matrix A is symmetric. Fix an indeterminate v . For any $n \in \mathbb{Z}$, set $[n]_v = \frac{v^n - v^{-n}}{v - v^{-1}} \in \mathbb{Q}(v)$. Let $[0]_v! = 1$ and $[n]_v! = [n]_v [n-1]_v \cdots [1]_v$ for any $n \in \mathbb{Z}_{>0}$.

The quantum group \mathbf{U} associated with a Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ is an associative algebra over $\mathbb{Q}(v)$ with unit element $\mathbf{1}$, generated by the elements E_i , $F_i (i \in I)$ and $K_\mu (\mu \in P^\vee)$ subject to the following relations

$$K_0 = \mathbf{1}, \quad K_\mu K_{\mu'} = K_{\mu + \mu'} \quad \text{for all } \mu, \mu' \in P^\vee;$$

$$K_\mu E_i K_{-\mu} = v^{\alpha_i(\mu)} E_i \quad \text{for all } i \in I, \mu \in P^\vee;$$

$$K_\mu F_i K_{-\mu} = v^{-\alpha_i(\mu)} F_i \quad \text{for all } i \in I, \mu \in P^\vee;$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_{-i}}{v - v^{-1}} \quad \text{for all } i, j \in I;$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k E_i^{(k)} E_j E_i^{(1-a_{ij}-k)} = 0 \quad \text{for all } i \neq j \in I;$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k F_i^{(k)} F_j F_i^{(1-a_{ij}-k)} = 0 \text{ for all } i \neq j \in I.$$

Here, $K_i = K_{h_i}$ and $E_i^{(n)} = E_i^n/[n]_v!$, $F_i^{(n)} = F_i^n/[n]_v!$.

Let \mathbf{U}^+ (resp. \mathbf{U}^-) be the subalgebra of \mathbf{U} generated by E_i (resp. F_i) for all $i \in I$, and \mathbf{U}^0 be the subalgebra of \mathbf{U} generated by K_μ for all $\mu \in P^\vee$. The quantum group \mathbf{U} has the following triangular decomposition

$$\mathbf{U} \cong \mathbf{U}^- \otimes \mathbf{U}^0 \otimes \mathbf{U}^+.$$

Let \mathbf{f} be the associative algebra defined by Lusztig in [16]. The algebra \mathbf{f} is generated by $\theta_i (i \in I)$ subject to the following relations

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \theta_i^{(k)} \theta_j \theta_i^{(1-a_{ij}-k)} = 0 \text{ for all } i \neq j \in I,$$

where $\theta_i^{(n)} = \theta_i^n/[n]_v!$.

There are two well-defined $\mathbb{Q}(v)$ -algebra homomorphisms $^+ : \mathbf{f} \rightarrow \mathbf{U}$ and $^- : \mathbf{f} \rightarrow \mathbf{U}$ satisfying $E_i = \theta_i^+$ and $F_i = \theta_i^-$ for all $i \in I$. The images of $^+$ and $^-$ are \mathbf{U}^+ and \mathbf{U}^- respectively.

2.2. Lusztig's symmetries. Corresponding to $i \in I$, Lusztig introduced the Lusztig's symmetry $T_i : \mathbf{U} \rightarrow \mathbf{U}$ ([10, 12, 16]). The formulas of T_i on the generators are:

$$\begin{aligned} (1) \quad & T_i(E_i) = -F_i K_i, \quad T_i(F_i) = -K_{-i} E_i; \\ & T_i(E_j) = \sum_{r+s=-a_{ij}} (-1)^r v^{-r} E_i^{(s)} E_j E_i^{(r)} \text{ for } i \neq j \in I; \\ (2) \quad & T_i(F_j) = \sum_{r+s=-a_{ij}} (-1)^r v^r F_i^{(r)} F_j F_i^{(s)} \text{ for } i \neq j \in I; \\ & T_i(K_\mu) = K_{\mu - \alpha_i(\mu)h_i}. \end{aligned}$$

Lusztig introduced two subalgebras ${}_i\mathbf{f}$ and ${}^i\mathbf{f}$ of \mathbf{f} . For any $j \in I$, $i \neq j$, $m \in \mathbb{N}$, define

$$f(i, j; m) = \sum_{r+s=m} (-1)^r v^{-r(-a_{ij}-m+1)} \theta_i^{(r)} \theta_j \theta_i^{(s)} \in \mathbf{f},$$

and

$$f'(i, j; m) = \sum_{r+s=m} (-1)^r v^{-r(-a_{ij}-m+1)} \theta_i^{(s)} \theta_j \theta_i^{(r)} \in \mathbf{f}.$$

The subalgebras ${}_i\mathbf{f}$ and ${}^i\mathbf{f}$ are generated by $f(i, j; m)$ and $f'(i, j; m)$ respectively.

Note that ${}_i\mathbf{f} = \{x \in \mathbf{f} \mid T_i(x^+) \in \mathbf{U}^+\}$ and ${}^i\mathbf{f} = \{x \in \mathbf{f} \mid T_i^{-1}(x^+) \in \mathbf{U}^+\}$ ([16]). Hence there exists a unique $T_i : {}_i\mathbf{f} \rightarrow {}^i\mathbf{f}$ such that $T_i(x^+) = T_i(x)^+$. Lusztig also showed that \mathbf{f} has the following direct sum decompositions $\mathbf{f} = {}_i\mathbf{f} \oplus \theta_i \mathbf{f} = {}^i\mathbf{f} \oplus \mathbf{f} \theta_i$. Denote by ${}_i\pi : \mathbf{f} \rightarrow {}_i\mathbf{f}$ and ${}^i\pi : \mathbf{f} \rightarrow {}^i\mathbf{f}$ the natural projections.

Lusztig also proved the following formulas.

Proposition 2.1 ([16]). *For any $-a_{ij} \geq m \in \mathbb{N}$, $T_i(f(i, j; m)) = f'(i, j; -a_{ij} - m)$.*

The formulas (1) and (2) are two special cases of Proposition 2.1.

3. GEOMETRIC REALIZATIONS

3.1. Geometric realization and canonical basis of \mathbf{f} . In this subsection, we shall review the geometric realization of \mathbf{f} introduced by Lusztig ([11, 13, 16]).

A quiver $Q = (I, H, s, t)$ consists of a vertex set I , an arrow set H , and two maps $s, t : H \rightarrow I$ such that an arrow $\rho \in H$ starts at $s(\rho)$ and terminates at $t(\rho)$. Let $h_{ij} = \#\{i \rightarrow j\}$, $a_{ij} = h_{ij} + h_{ji}$ and \mathbf{f} be the Lusztig's algebra corresponding to $A = (a_{ij})$. Let p be a prime and q be a power of p . Denote by \mathbb{F}_q the finite field with q elements and $\mathbb{K} = \overline{\mathbb{F}_q}$.

For a finite dimensional I -graded \mathbb{K} -vector space $\mathbf{V} = \bigoplus_{i \in I} V_i$, define

$$E_{\mathbf{V}} = \bigoplus_{\rho \in H} \text{Hom}_{\mathbb{K}}(V_{s(\rho)}, V_{t(\rho)}).$$

The dimension vector of \mathbf{V} is defined as $\underline{\dim} \mathbf{V} = \sum_{i \in I} (\dim_{\mathbb{K}} V_i) i \in \mathbb{N}I$. The algebraic group $G_{\mathbf{V}} = \prod_{i \in I} GL_{\mathbb{K}}(V_i)$ acts on $E_{\mathbf{V}}$ naturally.

Fix a nonzero element $\nu \in \mathbb{N}I$. Let

$$Y_{\nu} = \{\mathbf{y} = (\mathbf{i}, \mathbf{a}) \mid \sum_{l=1}^k a_l i_l = \nu\},$$

where $\mathbf{i} = (i_1, i_2, \dots, i_k)$, $i_l \in I$, $\mathbf{a} = (a_1, a_2, \dots, a_k)$, $a_l \in \mathbb{N}$, and

$$I^{\nu} = \{\mathbf{i} = (i_1, i_2, \dots, i_k) \mid \sum_{l=1}^k i_l = \nu\}.$$

Fix a finite dimensional I -graded \mathbb{K} -vector space \mathbf{V} such that $\underline{\dim} \mathbf{V} = \nu$. For any element $\mathbf{y} = (\mathbf{i}, \mathbf{a})$, a flag of type \mathbf{y} in \mathbf{V} is a sequence

$$\phi = (\mathbf{V} = \mathbf{V}^k \supset \mathbf{V}^{k-1} \supset \dots \supset \mathbf{V}^0 = 0)$$

of I -graded \mathbb{K} -vector spaces such that $\underline{\dim} \mathbf{V}^l / \mathbf{V}^{l-1} = a_l i_l$. Let $F_{\mathbf{y}}$ be the variety of all flags of type \mathbf{y} in \mathbf{V} . For any $x \in E_{\mathbf{V}}$, a flag ϕ is called x -stable if $x_{\rho}(V_{s(\rho)}^l) \subset V_{t(\rho)}^l$ for all l and all $\rho \in H$. Let

$$\tilde{F}_{\mathbf{y}} = \{(x, \phi) \in E_{\mathbf{V}} \times F_{\mathbf{y}} \mid \phi \text{ is } x\text{-stable}\}$$

and $\pi_{\mathbf{y}} : \tilde{F}_{\mathbf{y}} \rightarrow E_{\mathbf{V}}$ be the projection to $E_{\mathbf{V}}$.

Let $\bar{\mathbb{Q}}_l$ be the l -adic field and $\mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})$ be the bounded $G_{\mathbf{V}}$ -equivariant derived category of complexes of l -adic sheaves on $E_{\mathbf{V}}$. For each $\mathbf{y} \in Y_{\nu}$, $\mathcal{L}_{\mathbf{y}} = (\pi_{\mathbf{y}})_!(\mathbf{1}_{\tilde{F}_{\mathbf{y}}})[d_{\mathbf{y}}] \in \mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})$ is a semisimple perverse sheaf, where $d_{\mathbf{y}} = \dim \tilde{F}_{\mathbf{y}}$. Let $\mathcal{P}_{\mathbf{V}}$ be the set of isomorphism classes of simple perverse sheaves \mathcal{L} on $E_{\mathbf{V}}$ such that $\mathcal{L}[r]$ appears as a direct summand of $\mathcal{L}_{\mathbf{i}}$ for some $\mathbf{i} \in I^{\nu}$ and $r \in \mathbb{Z}$. Let $\mathcal{Q}_{\mathbf{V}}$ be the full subcategory of $\mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})$ consisting of all complexes which are isomorphic to finite direct sums of complexes in the set $\{\mathcal{L}[r] \mid \mathcal{L} \in \mathcal{P}_{\mathbf{V}}, r \in \mathbb{Z}\}$.

Let $K(\mathcal{Q}_{\mathbf{V}})$ be the Grothendieck group of $\mathcal{Q}_{\mathbf{V}}$. Define

$$v^{\pm}[\mathcal{L}] = [\mathcal{L}[\pm 1](\pm \frac{1}{2})],$$

where $\mathcal{L}(d)$ is the Tate twist of \mathcal{L} . Then, $K(\mathcal{Q}_{\mathbf{V}})$ is a free \mathcal{A} -module, where $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. Define

$$K(\mathcal{Q}) = \bigoplus_{\nu \in \mathbb{N}I} K(\mathcal{Q}_{\mathbf{V}}).$$

For $\nu, \nu', \nu'' \in \mathbb{N}I$ such that $\nu = \nu' + \nu''$ and three I -graded \mathbb{K} -vector spaces $\mathbf{V}, \mathbf{V}', \mathbf{V}''$ such that $\underline{\dim} \mathbf{V} = \nu$, $\underline{\dim} \mathbf{V}' = \nu'$, $\underline{\dim} \mathbf{V}'' = \nu''$, Lusztig constructed a functor

$$* : \mathcal{Q}_{\mathbf{V}'} \times \mathcal{Q}_{\mathbf{V}''} \rightarrow \mathcal{Q}_{\mathbf{V}}.$$

This functor induces an associative \mathcal{A} -bilinear multiplication

$$\begin{aligned} \circledast : K(\mathcal{Q}_{\mathbf{V}'}) \times K(\mathcal{Q}_{\mathbf{V}''}) &\rightarrow K(\mathcal{Q}_{\mathbf{V}}) \\ ([\mathcal{L}'], [\mathcal{L}'']) &\mapsto [\mathcal{L}'] \circledast [\mathcal{L}''] = [\mathcal{L}' * \mathcal{L}''] \end{aligned}$$

where $\mathcal{L}' * \mathcal{L}'' = (\mathcal{L}' * \mathcal{L}'')[m_{\nu'\nu''}](\frac{m_{\nu'\nu''}}{2})$ and $m_{\nu'\nu''} = \sum_{\rho \in H} \nu'_{s(\rho)} \nu''_{t(\rho)} - \sum_{i \in I} \nu'_i \nu''_i$. Then $K(\mathcal{Q})$ becomes an associative \mathcal{A} -algebra and the set $\{[\mathcal{L}] \mid \mathcal{L} \in \mathcal{P}_{\mathbf{V}}\}$ is a basis of $K(\mathcal{Q}_{\mathbf{V}})$.

Theorem 3.1 ([13]). *There is a unique \mathcal{A} -algebra isomorphism*

$$\lambda_{\mathcal{A}} : K(\mathcal{Q}) \rightarrow \mathbf{f}_{\mathcal{A}}$$

such that $\lambda_{\mathcal{A}}(\mathcal{L}_{\mathbf{y}}) = \theta_{\mathbf{y}}$ for all $\mathbf{y} \in Y_{\nu}$, where $\theta_{\mathbf{y}} = \theta_{i_1}^{(a_1)} \theta_{i_2}^{(a_2)} \dots \theta_{i_k}^{(a_k)}$ and $\mathbf{f}_{\mathcal{A}}$ is the integral form of \mathbf{f} .

Let $\mathbf{B}_{\nu} = \{\mathbf{b}_{\mathcal{L}} = \lambda_{\mathcal{A}}([\mathcal{L}]) \mid \mathcal{L} \in \mathcal{P}_{\mathbf{V}}\}$ and $\mathbf{B} = \sqcup_{\nu \in \mathbb{N}I} \mathbf{B}_{\nu}$. Then \mathbf{B} is the canonical basis of \mathbf{f} introduced by Lusztig in [11, 13].

3.2. Geometric realizations of ${}_i\mathbf{f}$ and ${}^i\mathbf{f}$. Assume that $i \in I$ is a sink. Let \mathbf{V} be a finite dimensional I -graded \mathbb{K} -vector space such that $\underline{\dim} \mathbf{V} = \nu$. Consider a subvariety ${}_iE_{\mathbf{V}}$ of $E_{\mathbf{V}}$

$${}_iE_{\mathbf{V}} = \{x \in E_{\mathbf{V}} \mid \bigoplus_{h \in H, t(h)=i} x_h : \bigoplus_{h \in H, t(h)=i} V_{s(h)} \rightarrow V_i \text{ is surjective}\}.$$

Let $j_{\mathbf{V}} : {}_iE_{\mathbf{V}} \rightarrow E_{\mathbf{V}}$ be the canonical embedding. For any $\mathbf{y} = (\mathbf{i}, \mathbf{a}) \in Y_{\nu}$, let

$${}_i\tilde{F}_{\mathbf{y}} = \{(x, \phi) \in {}_iE_{\mathbf{V}} \times F_{\mathbf{y}} \mid \phi \text{ is } x\text{-stable}\}$$

and ${}_i\pi_{\mathbf{y}} : {}_i\tilde{F}_{\mathbf{y}} \rightarrow {}_iE_{\mathbf{V}}$ be the projection to ${}_iE_{\mathbf{V}}$.

For any $\mathbf{y} \in Y_{\nu}$, ${}_i\mathcal{L}_{\mathbf{y}} = ({}_i\pi_{\mathbf{y}})_!(\mathbf{1}_{{}_i\tilde{F}_{\mathbf{y}}})[d_{\mathbf{y}}] \in \mathcal{D}_{G_{\mathbf{V}}}({}_iE_{\mathbf{V}})$ is a semisimple perverse sheaf. Let ${}_i\mathcal{P}_{\mathbf{V}}$ be the set of isomorphism classes of simple perverse sheaves \mathcal{L} on ${}_iE_{\mathbf{V}}$ such that $\mathcal{L}[r]$ appears as a direct summand of ${}_i\mathcal{L}_{\mathbf{i}}$ for some $\mathbf{i} \in I^{\nu}$ and $r \in \mathbb{Z}$. Let ${}_i\mathcal{Q}_{\mathbf{V}}$ be the full subcategory of $\mathcal{D}_{G_{\mathbf{V}}}({}_iE_{\mathbf{V}})$ consisting of all complexes which are isomorphic to finite direct sums of complexes in the set $\{\mathcal{L}[r] \mid \mathcal{L} \in {}_i\mathcal{P}_{\mathbf{V}}, r \in \mathbb{Z}\}$.

Let $K({}_i\mathcal{Q}_{\mathbf{V}})$ be the Grothendieck group of ${}_i\mathcal{Q}_{\mathbf{V}}$ and

$$K({}_i\mathcal{Q}) = \bigoplus_{[V]} K({}_i\mathcal{Q}_{\mathbf{V}}).$$

Naturally, we have two functors $j_{\mathbf{V}}! : \mathcal{D}_{G_{\mathbf{V}}}({}_iE_{\mathbf{V}}) \rightarrow \mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})$ and $j_{\mathbf{V}}^* : \mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}}) \rightarrow \mathcal{D}_{G_{\mathbf{V}}}({}_iE_{\mathbf{V}})$.

For any $\mathbf{y} \in Y_{\nu}$, we have the following fiber product

$$\begin{array}{ccc} {}_i\tilde{F}_{\mathbf{y}} & \xrightarrow{\tilde{j}_{\mathbf{V}}} & \tilde{F}_{\mathbf{y}} \\ \downarrow {}_i\pi_{\mathbf{y}} & & \downarrow \pi_{\mathbf{y}} \\ {}_iE_{\mathbf{V}} & \xrightarrow{j_{\mathbf{V}}} & E_{\mathbf{V}} \end{array}$$

So

$$(3) \quad j_{\mathbf{V}}^*\mathcal{L}_{\mathbf{y}} = j_{\mathbf{V}}^*(\pi_{\mathbf{y}})_!(\mathbf{1}_{\tilde{F}_{\mathbf{y}}})[d_{\mathbf{y}}] = ({}_i\pi_{\mathbf{y}})_!\tilde{j}_{\mathbf{V}}^*(\mathbf{1}_{\tilde{F}_{\mathbf{y}}})[d_{\mathbf{y}}] = ({}_i\pi_{\mathbf{y}})_!(\mathbf{1}_{{}_i\tilde{F}_{\mathbf{y}}})[d_{\mathbf{y}}] = {}_i\mathcal{L}_{\mathbf{y}}.$$

That is $j_{\mathbf{V}}^*({}_i\mathcal{Q}_{\mathbf{V}}) = {}_i\mathcal{Q}_{\mathbf{V}}$. Hence $j_{\mathbf{V}}^* : \mathcal{Q}_{\mathbf{V}} \rightarrow {}_i\mathcal{Q}_{\mathbf{V}}$ and $j^* : K(\mathcal{Q}) \rightarrow K({}_i\mathcal{Q})$ can be defined.

Consider the following diagram

$$(4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \theta_i \mathbf{f}_{\mathcal{A}} & \xrightarrow{i} & \mathbf{f}_{\mathcal{A}} & \xrightarrow{{}_i\pi_{\mathcal{A}}} & {}_i\mathbf{f}_{\mathcal{A}} \longrightarrow 0 \\ & & & & \downarrow \lambda'_{\mathcal{A}} & & \downarrow \text{dotted} \\ & & & & K(\mathcal{Q}) & \xrightarrow{j^*} & K({}_i\mathcal{Q}) \longrightarrow 0 \end{array}$$

where $\lambda'_{\mathcal{A}}$ is the inverse of $\lambda_{\mathcal{A}}$. Since $j^* \circ \lambda'_{\mathcal{A}} \circ i = 0$, there exists a map ${}_i\lambda'_{\mathcal{A}} : {}_i\mathbf{f}_{\mathcal{A}} \rightarrow K({}_i\mathcal{Q})$ such that the above diagram (4) commutes.

Proposition 3.2. *The map ${}_i\lambda'_{\mathcal{A}} : {}_i\mathbf{f}_{\mathcal{A}} \rightarrow K({}_i\mathcal{Q})$ is an isomorphism of \mathcal{A} -algebras.*

The proof of Proposition 3.2 will be given in Section 4.2.

Assume that $i \in I$ is a source. We can give a geometric realization of ${}_i\mathbf{f}$ similarly. Consider a subvariety ${}_iE_{\mathbf{V}}$ of $E_{\mathbf{V}}$

$${}_iE_{\mathbf{V}} = \{x \in E_{\mathbf{V}} \mid \bigoplus_{h \in H, s(h)=i} x_h : V_i \rightarrow \bigoplus_{h \in H, s(h)=i} V_{t(h)} \text{ is injective}\}.$$

Let $j_{\mathbf{V}} : {}_iE_{\mathbf{V}} \rightarrow E_{\mathbf{V}}$ be the canonical embedding. The definitions of ${}_i\mathcal{Q}_{\mathbf{V}}$, $K({}_i\mathcal{Q}_{\mathbf{V}})$ and $K({}_i\mathcal{Q})$ are similar to those of ${}_i\mathcal{Q}_{\mathbf{V}}$, $K({}_i\mathcal{Q}_{\mathbf{V}})$ and $K({}_i\mathcal{Q})$ respectively. We can also define $j_{\mathbf{V}}^* : \mathcal{Q}_{\mathbf{V}} \rightarrow {}_i\mathcal{Q}_{\mathbf{V}}$, $j^* : K(\mathcal{Q}) \rightarrow K({}_i\mathcal{Q})$ and ${}_i\lambda'_{\mathcal{A}} : {}_i\mathbf{f}_{\mathcal{A}} \rightarrow K({}_i\mathcal{Q})$.

Similarly to Proposition 3.2, we have the following proposition.

Proposition 3.3. *The map ${}_i\lambda'_{\mathcal{A}} : {}_i\mathbf{f}_{\mathcal{A}} \rightarrow K({}_i\mathcal{Q})$ is an isomorphism of \mathcal{A} -algebras.*

□

3.3. Geometric realization of $T_i : {}_i\mathbf{f} \rightarrow {}^i\mathbf{f}$. Assume that i is a sink of $Q = (I, H, s, t)$. So i is a source of $Q' = \sigma_i Q = (I, H', s, t)$, where $\sigma_i Q$ is the quiver by reversing the directions of all arrows in Q containing i . For any $\nu, \nu' \in \mathbb{N}I$ such that $\nu' = s_i \nu$ and I -graded \mathbb{K} -vector spaces \mathbf{V}, \mathbf{V}' such that $\underline{\dim} \mathbf{V} = \nu$, $\underline{\dim} \mathbf{V}' = \nu'$, consider the following correspondence ([15, 5])

$$(5) \quad {}_i E_{\mathbf{V}, Q} \xleftarrow{\alpha} Z_{\mathbf{V}\mathbf{V}'} \xrightarrow{\beta} {}^i E_{\mathbf{V}', Q'},$$

where

- (1) $Z_{\mathbf{V}\mathbf{V}'}$ is the subset in $E_{\mathbf{V}, Q} \times E_{\mathbf{V}', Q'}$ consisting of all (x, y) satisfying the following conditions
 - (a) for any $h \in H$ such that $t(h) \neq i$ and $h \in H'$, $x_h = y_h$;
 - (b) the following sequence is exact

$$0 \longrightarrow V_i' \xrightarrow{\bigoplus_{h \in H', s(h)=i} y_h} \bigoplus_{h \in H, t(h)=i} V_{s(h)} \xrightarrow{\bigoplus_{h \in H, t(h)=i} x_h} V_i \longrightarrow 0$$

- (2) $\alpha(x, y) = x$ and $\beta(x, y) = y$.

From now on, ${}_i E_{\mathbf{V}, Q}$ is denoted by ${}_i E_{\mathbf{V}}$ and ${}^i E_{\mathbf{V}', Q'}$ is denoted by ${}^i E_{\mathbf{V}'}$. Let

$$G_{\mathbf{V}\mathbf{V}'} = GL(V_i) \times GL(V_i') \times \prod_{j \neq i} GL(V_j) \cong GL(V_i) \times GL(V_i') \times \prod_{j \neq i} GL(V_j'),$$

which acts on $Z_{\mathbf{V}\mathbf{V}'}$ naturally.

By (5), we have

$$(6) \quad \mathcal{D}_{G_{\mathbf{V}}}({}_i E_{\mathbf{V}}) \xrightarrow{\alpha^*} \mathcal{D}_{G_{\mathbf{V}\mathbf{V}'}}(Z_{\mathbf{V}\mathbf{V}'}) \xleftarrow{\beta^*} \mathcal{D}_{G_{\mathbf{V}'}}({}^i E_{\mathbf{V}'}).$$

Since α and β are principal bundles with fibers $\text{Aut}(V_i')$ and $\text{Aut}(V_i)$ respectively, α^* and β^* are equivalences of categories by Section 2.2.5 in [1]. Hence, for any $\mathcal{L} \in \mathcal{D}_{G_{\mathbf{V}}}({}_i E_{\mathbf{V}})$ there exists a unique $\mathcal{L}' \in \mathcal{D}_{G_{\mathbf{V}'}}({}^i E_{\mathbf{V}'})$ such that $\alpha^*(\mathcal{L}) = \beta^*(\mathcal{L}')$. Define

$$\begin{aligned} \tilde{\omega}_i : \mathcal{D}_{G_{\mathbf{V}}}({}_i E_{\mathbf{V}}) &\rightarrow \mathcal{D}_{G_{\mathbf{V}'}}({}^i E_{\mathbf{V}'}), \\ \mathcal{L} &\mapsto \mathcal{L}'[-s(\mathbf{V})](-\frac{s(\mathbf{V})}{2}) \end{aligned}$$

where $s(\mathbf{V}) = \dim GL(V_i) - \dim GL(V_i')$. Since α^* and β^* are equivalences of categories, $\tilde{\omega}_i$ is also an equivalence of categories.

Proposition 3.4. *It holds that $\tilde{\omega}_i({}_i \mathcal{Q}_{\mathbf{V}}) = {}^i \mathcal{Q}_{\mathbf{V}'}$.*

The proof of Proposition 3.4 will be given in Section 4.3.

Hence, we can define $\tilde{\omega}_i : {}_i \mathcal{Q}_{\mathbf{V}} \rightarrow {}^i \mathcal{Q}_{\mathbf{V}'}$ and $\tilde{\omega}_i : K({}_i \mathcal{Q}) \rightarrow K({}^i \mathcal{Q})$. We have the following theorem.

Theorem 3.5. *We have the following commutative diagram*

$$\begin{array}{ccc} {}_i\mathbf{f}_{\mathcal{A}} & \xrightarrow{T_i} & {}^i\mathbf{f}_{\mathcal{A}} \\ \downarrow {}_i\lambda'_{\mathcal{A}} & & \downarrow {}^i\lambda'_{\mathcal{A}} \\ K({}_i\mathcal{Q}) & \xrightarrow{\tilde{\omega}_i} & K({}^i\mathcal{Q}) \end{array}$$

The proof of Theorem 3.5 will be given in Section 4.3.

3.4. $T_i : {}_i\mathbf{f} \rightarrow {}^i\mathbf{f}$ and canonical bases. In [14], Lusztig showed that Lusztig's symmetries and canonical bases are compatible. In this section, we shall give a geometric interpretation of this result by using the geometric realization of T_i .

Let \mathbf{B} be the canonical basis of \mathbf{f} . Since $\theta_i\mathbf{f}$ is the kernel of ${}_i\pi : \mathbf{f} \rightarrow {}_i\mathbf{f}$ and $\mathbf{B} \cap \theta_i\mathbf{f}$ is a $\mathbb{Q}(v)$ -basis of $\theta_i\mathbf{f}$, ${}_i\mathbf{B} = {}_i\pi(\mathbf{B})$ is a $\mathbb{Q}(v)$ -basis of ${}_i\mathbf{f}$. Similarly, ${}^i\mathbf{B} = {}^i\pi(\mathbf{B})$ is a $\mathbb{Q}(v)$ -basis of ${}^i\mathbf{f}$.

Lusztig proved the following theorem.

Theorem 3.6 ([14]). *Lusztig's symmetry $T_i : {}_i\mathbf{f} \rightarrow {}^i\mathbf{f}$ maps any element of ${}_i\mathbf{B}$ to an element of ${}^i\mathbf{B}$. Thus, there exists a unique bijection $\kappa_i : \mathbf{B} - \mathbf{B} \cap \theta_i\mathbf{f} \rightarrow \mathbf{B} - \mathbf{B} \cap \mathbf{f}\theta_i$ such that $T_i({}_i\pi(b)) = {}^i\pi(\kappa_i(b))$.*

Let i be a sink of a quiver Q . So i is a source of $Q' = \sigma_i Q$. By Theorem 3.1, Proposition 3.3, the formula (3) and the commutative diagram (4), we have

$$(7) \quad {}_i\mathbf{B} = \sqcup_{\nu \in \mathbb{N}I} \{\mathbf{b}_{\mathcal{L}} = {}_i\lambda_{\mathcal{A}}([\mathcal{L}]) \mid \mathcal{L} \in {}_i\mathcal{P}_{\mathbf{V}}, \underline{\dim}\mathbf{V} = \nu\}.$$

Similarly, we have

$$(8) \quad {}^i\mathbf{B} = \sqcup_{\nu' \in \mathbb{N}I} \{\mathbf{b}_{\mathcal{L}} = {}^i\lambda_{\mathcal{A}}([\mathcal{L}]) \mid \mathcal{L} \in {}^i\mathcal{P}_{\mathbf{V}'}, \underline{\dim}\mathbf{V}' = \nu'\}.$$

Fix any $\nu, \nu' \in \mathbb{N}I$ such that $\nu' = s_i\nu$ and I -graded \mathbb{K} -vector spaces \mathbf{V}, \mathbf{V}' such that $\underline{\dim}\mathbf{V} = \nu, \underline{\dim}\mathbf{V}' = \nu'$.

In (6), the functors α^* and β^* are equivalences of categories. Hence the functor

$$\tilde{\omega}_i : {}_i\mathcal{Q}_{\mathbf{V}} \rightarrow {}^i\mathcal{Q}_{\mathbf{V}'}$$

maps any simple perverse sheaf in ${}_i\mathcal{Q}_{\mathbf{V}}$ to a simple perverse sheaf in ${}^i\mathcal{Q}_{\mathbf{V}'}$. That is, $\tilde{\omega}_i({}_i\mathcal{P}_{\mathbf{V}}) = {}^i\mathcal{P}_{\mathbf{V}'}$. So the map

$$\tilde{\omega}_i : K({}_i\mathcal{Q}) \rightarrow K({}^i\mathcal{Q})$$

satisfies

$$\tilde{\omega}_i(\{[\mathcal{L}] \mid \mathcal{L} \in {}_i\mathcal{P}_{\mathbf{V}}\}) = \{[\mathcal{L}] \mid \mathcal{L} \in {}^i\mathcal{P}_{\mathbf{V}'}\}.$$

By Theorem 3.5, (7) and (8), it holds that $T_i({}_i\mathbf{B}) = {}^i\mathbf{B}$ and we get a geometric interpretation of Theorem 3.6.

4. HALL ALGEBRA APPROACHES

4.1. Hall algebra approach to \mathbf{f} . In this subsection, we shall review the Hall algebra approach to \mathbf{f} ([18, 15, 8, 9]).

Let $Q = (I, H, s, t)$ be a quiver. In Section 3.1, $E_{\mathbf{V}}$ and $G_{\mathbf{V}}$ are defined for any I -graded \mathbb{K} -vector space \mathbf{V} . Let F^n be the Frobenius morphism. The sets $E_{\mathbf{V}}^{F^n}$ and $G_{\mathbf{V}}^{F^n}$ consist of the F^n -fixed points in $E_{\mathbf{V}}$ and $G_{\mathbf{V}}$ respectively.

Lusztig defined $\underline{\mathcal{F}}_{\mathbf{V}}^n$ as the set of all $G_{\mathbf{V}}^{F^n}$ -invariant $\bar{\mathbb{Q}}_l$ -functions on $E_{\mathbf{V}}^{F^n}$ and we can give a multiplication on $\underline{\mathcal{F}}^n = \bigoplus_{\nu \in \mathbb{N}I} \underline{\mathcal{F}}_{\mathbf{V}}^n$ to obtain the Hall algebra. For any $i \in I$, let \mathbf{V}_i be the I -graded \mathbb{K} -vector space with dimension vector i and f_i be the constant function on $E_{\mathbf{V}_i}^{F^n}$ with value 1. Denote by \mathcal{F}^n the composition subalgebra of $\underline{\mathcal{F}}^n$ generated by f_i and $\mathcal{F}_{\mathbf{V}}^n = \underline{\mathcal{F}}_{\mathbf{V}}^n \cap \mathcal{F}^n$. Let $\mathcal{F} = \bigoplus_{\nu \in \mathbb{N}I} \mathcal{F}_{\mathbf{V}}$ be the generic form of \mathcal{F}^n and $\mathcal{F}_{\mathcal{A}}$ be the integral form of \mathcal{F} ([15]).

Theorem 4.1 ([18, 15]). *There exists an isomorphism of \mathcal{A} -algebras*

$$\varpi_{\mathcal{A}} : \mathbf{f}_{\mathcal{A}} \rightarrow \mathcal{F}_{\mathcal{A}}$$

such that $\varpi_{\mathcal{A}}(\theta_i) = f_i$.

For any $\mathcal{L} \in \mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})$, there is a function $\chi_{\mathcal{L}}^n : E_{\mathbf{V}}^{F^n} \rightarrow \bar{\mathbb{Q}}_l$ (Section I.2.12 in [7]). Hence, we have the following map

$$\begin{aligned} \chi^n : \mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}}) &\rightarrow \underline{\mathcal{F}}_{\mathbf{V}}^n \\ \mathcal{L} &\mapsto \chi_{\mathcal{L}}^n \end{aligned}$$

The restriction of this map on the subcategory $\mathcal{Q}_{\mathbf{V}}$ is also denoted by

$$\chi^n : \mathcal{Q}_{\mathbf{V}} \rightarrow \underline{\mathcal{F}}_{\mathbf{V}}^n.$$

Lusztig proved that $\chi^n(\mathcal{Q}_{\mathbf{V}}) \subset \mathcal{F}_{\mathbf{V}}^n$ in [15]. Hence, we can define $\chi^n : \mathcal{Q}_{\mathbf{V}} \rightarrow \mathcal{F}_{\mathbf{V}}^n$, which induces $\chi : \mathcal{Q}_{\mathbf{V}} \rightarrow \mathcal{F}_{\mathbf{V}}$ naturally. Hence, we get a map $\chi_{\mathcal{A}} : K(\mathcal{Q}) \rightarrow \mathcal{F}_{\mathcal{A}}$.

Lusztig proved the following proposition.

Proposition 4.2 ([15]). *$\chi_{\mathcal{A}} : K(\mathcal{Q}) \rightarrow \mathcal{F}_{\mathcal{A}}$ is an isomorphism of \mathcal{A} -algebras such that $\chi_{\mathcal{A}}([\mathcal{L}_i]) = f_i$ and the following diagram is commutative*

$$\begin{array}{ccc} K(\mathcal{Q}) & \xrightarrow{\lambda_{\mathcal{A}}} & \mathbf{f}_{\mathcal{A}} \\ \downarrow \chi_{\mathcal{A}} & \nearrow \varpi_{\mathcal{A}} & \\ \mathcal{F}_{\mathcal{A}} & & \end{array}$$

4.2. Hall algebra approaches to ${}_i\mathbf{f}$ and ${}^i\mathbf{f}$. Let i be a sink of Q . In Section 3.2, ${}_iE_{\mathbf{V}}$ and $j_{\mathbf{V}} : {}_iE_{\mathbf{V}} \rightarrow E_{\mathbf{V}}$ are defined for any I -graded \mathbb{K} -vector space \mathbf{V} . Similarly, ${}_iE_{\mathbf{V}}^{F^n}$ is defined as the F^n -fixed points set in ${}_iE_{\mathbf{V}}$ and we have $j_{\mathbf{V}} : {}_iE_{\mathbf{V}}^{F^n} \rightarrow E_{\mathbf{V}}^{F^n}$.

Lusztig also defined ${}_i\underline{\mathcal{F}}_{\mathbf{V}}^n$ as the set of all $G_{\mathbf{V}}^{F^n}$ -invariant $\bar{\mathbb{Q}}_l$ -functions on ${}_iE_{\mathbf{V}}^{F^n}$. Similarly to the case in Section 4.1, the Hall algebra is denoted by ${}_i\underline{\mathcal{F}}^n = \bigoplus_{\nu \in \mathbb{N}I} {}_i\underline{\mathcal{F}}_{\mathbf{V}}^n$, the composition subalgebra is denoted by ${}_i\mathcal{F}^n = \bigoplus_{\nu \in \mathbb{N}I} {}_i\mathcal{F}_{\mathbf{V}}^n$ and the generic form is denoted by ${}_i\mathcal{F} := \bigoplus_{\nu \in \mathbb{N}I} {}_i\mathcal{F}_{\mathbf{V}}$.

Naturally, we have two maps $j_{\mathbf{V}}^* : \mathcal{F}_{\mathbf{V}} \rightarrow {}_i\mathcal{F}_{\mathbf{V}}$ and $j_{\mathbf{V}!} : {}_i\mathcal{F}_{\mathbf{V}} \rightarrow \mathcal{F}_{\mathbf{V}}$. Considering all dimension vectors, we have $j_! : {}_i\mathcal{F} \rightarrow \mathcal{F}$ and $j^* : \mathcal{F} \rightarrow {}_i\mathcal{F}$.

Proposition 4.3 ([15]). *We have the following commutative diagram*

$$\begin{array}{ccccc} {}_i\mathbf{f} & \longrightarrow & \mathbf{f} & \xrightarrow{i\pi} & {}_i\mathbf{f} \\ \cong \downarrow {}_i\varpi & & \cong \downarrow \varpi & & \cong \downarrow {}_i\varpi \\ {}_i\mathcal{F} & \xrightarrow{j_!} & \mathcal{F} & \xrightarrow{j^*} & {}_i\mathcal{F} \end{array}$$

where ${}_i\varpi$ is the isomorphism induced by ϖ .

Next, we shall prove Proposition 3.2.

For any $\mathcal{L} \in \mathcal{D}_{G_{\mathbf{V}}}({}_iE_{\mathbf{V}})$, there is also a function $\chi_{\mathcal{L}}^n : {}_iE_{\mathbf{V}}^{F^n} \rightarrow \bar{\mathbb{Q}}_l$. Hence, we have the following map

$$\begin{aligned} {}_i\chi^n : \mathcal{D}_{G_{\mathbf{V}}}({}_iE_{\mathbf{V}}) &\rightarrow {}_i\mathcal{F}_{\mathbf{V}}^n \\ \mathcal{L} &\mapsto \chi_{\mathcal{L}}^n \end{aligned}$$

The restriction of this map on the subcategory ${}_i\mathcal{Q}_{\mathbf{V}}$ is also denoted by

$${}_i\chi^n : {}_i\mathcal{Q}_{\mathbf{V}} \rightarrow {}_i\mathcal{F}_{\mathbf{V}}^n.$$

Proposition 4.4. *It holds that ${}_i\chi^n({}_i\mathcal{Q}_{\mathbf{V}}) \subset {}_i\mathcal{F}_{\mathbf{V}}^n$.*

Proof. By the properties of χ and ${}_i\chi$ (Theorem III.12.1(5) in [7]), we have the following commutative diagram

$$(9) \quad \begin{array}{ccc} \mathcal{Q}_{\mathbf{V}} & \xrightarrow{j_{\mathbf{V}}^*} & {}_i\mathcal{Q}_{\mathbf{V}} \\ \downarrow \chi^n & & \downarrow {}_i\chi^n \\ \mathcal{F}_{\mathbf{V}}^n & \xrightarrow{j_{\mathbf{V}}^*} & {}_i\mathcal{F}_{\mathbf{V}}^n \end{array}$$

By the commutative diagram (9), $j_{\mathbf{V}}^*(\mathcal{F}_{\mathbf{V}}^n) \subset {}_i\mathcal{F}_{\mathbf{V}}^n$ and $j_{\mathbf{V}}^*(\mathcal{Q}_{\mathbf{V}}) = {}_i\mathcal{Q}_{\mathbf{V}}$, we have ${}_i\chi^n({}_i\mathcal{Q}_{\mathbf{V}}) \subset {}_i\mathcal{F}_{\mathbf{V}}^n$. □

Hence, we can define ${}_i\chi^n : {}_i\mathcal{Q}_{\mathbf{V}} \rightarrow {}_i\mathcal{F}_{\mathbf{V}}^n$, which induces ${}_i\chi : {}_i\mathcal{Q}_{\mathbf{V}} \rightarrow {}_i\mathcal{F}_{\mathbf{V}}$ and ${}_i\chi_{\mathcal{A}} : K({}_i\mathcal{Q}) \rightarrow {}_i\mathcal{F}_{\mathcal{A}}$.

The commutative diagram (9) implies the following proposition.

Proposition 4.5. *We have the following commutative diagram*

$$\begin{array}{ccc} K(\mathcal{Q}) & \xrightarrow{j^*} & K({}_i\mathcal{Q}) \\ \downarrow \chi_{\mathcal{A}} & & \downarrow {}_i\chi_{\mathcal{A}} \\ \mathcal{F}_{\mathcal{A}} & \xrightarrow{j^*} & {}_i\mathcal{F}_{\mathcal{A}} \end{array}$$

□

Proof of Proposition 3.2. First, we shall prove the following commutative diagram

$$\begin{array}{ccc} {}_i\mathbf{f}_{\mathcal{A}} & \xrightarrow{i\varpi_{\mathcal{A}}} & {}_i\mathcal{F}_{\mathcal{A}} \\ \downarrow i\lambda'_{\mathcal{A}} & \nearrow i\chi_{\mathcal{A}} & \\ K({}_i\mathcal{Q}) & & \end{array}$$

Consider the following diagram

$$\begin{array}{ccc} \mathbf{f}_{\mathcal{A}} & \longrightarrow & {}_i\mathbf{f}_{\mathcal{A}} \\ \downarrow & & \downarrow \\ K(\mathcal{Q}) & \longrightarrow & K({}_i\mathcal{Q}) \\ \downarrow & & \downarrow \\ \mathcal{F}_{\mathcal{A}} & \longrightarrow & {}_i\mathcal{F}_{\mathcal{A}} \end{array}$$

Since three squares and the triangle in the left are commutative, the triangle in the right is also commutative.

Proposition 4.3 implies that $i\varpi_{\mathcal{A}} : {}_i\mathbf{f}_{\mathcal{A}} \rightarrow {}_i\mathcal{F}_{\mathcal{A}}$ is isomorphic. Hence $i\lambda'_{\mathcal{A}} : {}_i\mathbf{f}_{\mathcal{A}} \rightarrow K({}_i\mathcal{Q})$ is injective. The commutative diagram (4) in the definition of $i\lambda'_{\mathcal{A}}$ implies $i\lambda'_{\mathcal{A}} : {}_i\mathbf{f}_{\mathcal{A}} \rightarrow K({}_i\mathcal{Q})$ is surjection. Hence, $i\lambda'_{\mathcal{A}} : {}_i\mathbf{f}_{\mathcal{A}} \rightarrow K({}_i\mathcal{Q})$ is isomorphic. \square

In the proof, we get the following proposition.

Proposition 4.6. *We have the following commutative diagram*

$$\begin{array}{ccc} K({}_i\mathcal{Q}) & \xrightarrow{i\lambda_{\mathcal{A}}} & {}_i\mathbf{f}_{\mathcal{A}} \\ \downarrow i\chi_{\mathcal{A}} & \nearrow i\varpi_{\mathcal{A}} & \\ {}_i\mathcal{F}_{\mathcal{A}} & & \end{array}$$

where all maps are isomorphisms of \mathcal{A} -algebras and $i\lambda_{\mathcal{A}}$ is the inverse of $i\lambda'_{\mathcal{A}}$. \square

Assume that i is a source of \mathcal{Q} . The notations and results in this case are completely similar to the case that i is a sink. We can define ${}^i\mathcal{F}_{\mathbf{V}}^n$, ${}^i\mathcal{F}^n = \bigoplus_{\nu \in \mathbb{N}I} {}^i\mathcal{F}_{\mathbf{V}}^n$ and ${}^i\mathcal{F} = \bigoplus_{\nu \in \mathbb{N}I} {}^i\mathcal{F}_{\mathbf{V}}$. We also have two maps $j_{\mathbf{V}}^* : \mathcal{F}_{\mathbf{V}} \rightarrow {}^i\mathcal{F}_{\mathbf{V}}$ and $j_{\mathbf{V}!} : {}^i\mathcal{F}_{\mathbf{V}} \rightarrow \mathcal{F}_{\mathbf{V}}$. Considering all dimension vectors, we have $j_! : {}^i\mathcal{F} \rightarrow \mathcal{F}$ and $j^* : \mathcal{F} \rightarrow {}^i\mathcal{F}$.

Proposition 4.7 ([15]). *We have the following commutative diagram*

$$\begin{array}{ccccc} {}^i\mathbf{f} & \longrightarrow & \mathbf{f} & \xrightarrow{i\pi} & {}^i\mathbf{f} \\ \cong \downarrow i\varpi & & \cong \downarrow \varpi & & \cong \downarrow i\varpi \\ {}^i\mathcal{F} & \xrightarrow{j_!} & \mathcal{F} & \xrightarrow{j^*} & {}^i\mathcal{F} \end{array}$$

where ${}^i\varpi$ is the isomorphism induced by ϖ .

We can also define ${}^i\chi : {}^i\mathcal{Q}_{\mathbf{V}} \rightarrow {}^i\mathcal{F}_{\mathbf{V}}$ and ${}^i\chi_{\mathcal{A}} : K({}^i\mathcal{Q}) \rightarrow {}^i\mathcal{F}_{\mathcal{A}}$.

Proposition 4.8. *We have the following commutative diagram*

$$\begin{array}{ccc} K(\mathcal{Q}) & \xrightarrow{j^*} & K({}^i\mathcal{Q}) \\ \downarrow \chi_{\mathcal{A}} & & \downarrow {}^i\chi_{\mathcal{A}} \\ \mathcal{F}_{\mathcal{A}} & \xrightarrow{j^*} & {}^i\mathcal{F}_{\mathcal{A}} \end{array}$$

□

Proposition 4.9. *We have the following commutative diagram*

$$\begin{array}{ccc} K({}^i\mathcal{Q}) & \xrightarrow{{}^i\lambda_{\mathcal{A}}} & {}^i\mathbf{f}_{\mathcal{A}} \\ \downarrow {}^i\chi_{\mathcal{A}} & \nearrow {}^i\varpi_{\mathcal{A}} & \\ {}^i\mathcal{F}_{\mathcal{A}} & & \end{array}$$

where all maps are isomorphisms of \mathcal{A} -algebras and ${}^i\lambda_{\mathcal{A}}$ is the inverse of ${}^i\lambda'_{\mathcal{A}}$.

□

4.3. Hall algebra approach to $T_i : {}^i\mathbf{f} \rightarrow {}^i\mathbf{f}$ and the proof of Theorem 3.5.

Let i be a sink of a quiver $Q = (I, H, s, t)$. So i is a source of $Q' = \sigma_i Q = (I, H', s, t)$. For any ν and $\nu' \in \mathbb{N}I$ such that $\nu' = s_i \nu$, and two I -graded \mathbb{K} -vector spaces \mathbf{V} and \mathbf{V}' such that $\underline{\dim} \mathbf{V} = \nu$ and $\underline{\dim} \mathbf{V}' = \nu'$, the following correspondence is considered in Section 3.3

$${}^iE_{\mathbf{V}, Q} \xleftarrow{\alpha} Z_{\mathbf{V}\mathbf{V}'} \xrightarrow{\beta} {}^iE_{\mathbf{V}', Q'}.$$

Similarly, $Z_{\mathbf{V}\mathbf{V}'}^{F_n}$ is defined as the F_n -fixed points set in $Z_{\mathbf{V}\mathbf{V}'}$ and we have

$${}^iE_{\mathbf{V}, Q}^{F_n} \xleftarrow{\alpha} Z_{\mathbf{V}\mathbf{V}'}^{F_n} \xrightarrow{\beta} {}^iE_{\mathbf{V}', Q'}^{F_n}.$$

Note that α and β are principal bundles with fibers $\text{Aut}(V'_i)$ and $\text{Aut}(V_i)$ respectively. Hence, for any $f \in {}^i\mathcal{F}_{\mathbf{V}}^n$, there exists a unique $g \in {}^i\mathcal{F}_{\mathbf{V}'}^n$ such that $\alpha^*(f) = \beta^*(g)$. Define

$$\begin{aligned} \omega_i : {}^i\mathcal{F}_{\mathbf{V}}^n &\rightarrow {}^i\mathcal{F}_{\mathbf{V}'}^n \\ f &\mapsto (p^n)^{-\frac{s(\mathbf{V})}{2}} g \end{aligned}$$

Lusztig proved that $\omega_i({}^i\mathcal{F}_{\mathbf{V}}^n) \subset {}^i\mathcal{F}_{\mathbf{V}'}^n$. Hence, we have $\omega_i : {}^i\mathcal{F}_{\mathbf{V}}^n \rightarrow {}^i\mathcal{F}_{\mathbf{V}'}^n$ and $\omega_i : {}^i\mathcal{F}_{\mathbf{V}} \rightarrow {}^i\mathcal{F}_{\mathbf{V}'}$. Considering all dimension vectors, we have $\omega_i : {}^i\mathcal{F} \rightarrow {}^i\mathcal{F}$.

Lusztig proved the following theorem.

Theorem 4.10 ([15]). *We have the following commutative diagram*

$$\begin{array}{ccc} {}^i\mathbf{f} & \xrightarrow{T_i} & {}^i\mathbf{f} \\ \downarrow {}^i\varpi & & \downarrow {}^i\varpi \\ {}^i\mathcal{F} & \xrightarrow{\omega_i} & {}^i\mathcal{F} \end{array}$$

Proof of Proposition 3.4. By the properties of ${}^i\chi^n$ and ${}^i\chi^n$ (Theorem III.12.1(4,5) in [7]), we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{D}_{G_{\mathbf{V}}}({}^iE_{\mathbf{V}}) & \xrightarrow{\tilde{\omega}_i} & \mathcal{D}_{G_{\mathbf{V}'}}({}^iE_{\mathbf{V}'}) \\ \downarrow {}^i\chi^n & & \downarrow {}^i\chi^n \\ {}^i\mathcal{F}_{\mathbf{V}}^n & \xrightarrow{\omega_i^n} & {}^i\mathcal{F}_{\mathbf{V}'}^n \end{array}$$

Hence, we have

$$\begin{array}{ccc} \mathcal{D}_{G_{\mathbf{V}}}({}^iE_{\mathbf{V}}) & \xrightarrow{\tilde{\omega}_i} & \mathcal{D}_{G_{\mathbf{V}'}}({}^iE_{\mathbf{V}'}) \\ \downarrow \prod_{n \in \mathbb{Z}_{\geq 1}} {}^i\chi^n & & \downarrow \prod_{n \in \mathbb{Z}_{\geq 1}} {}^i\chi^n \\ \prod_{n \in \mathbb{Z}_{\geq 1}} {}^i\mathcal{F}_{\mathbf{V}}^n & \xrightarrow{\prod_{n \in \mathbb{Z}_{\geq 1}} \omega_i^n} & \prod_{n \in \mathbb{Z}_{\geq 1}} {}^i\mathcal{F}_{\mathbf{V}'}^n \end{array}$$

By Proposition 4.4, ${}^i\chi^n({}^i\mathcal{Q}_{\mathbf{V}}) \subset {}^i\mathcal{F}_{\mathbf{V}}^n$. Hence, we have

$$\begin{array}{ccc} {}^i\mathcal{Q}_{\mathbf{V}} & \xrightarrow{\tilde{\omega}_i} & \mathcal{D}_{G_{\mathbf{V}'}}({}^iE_{\mathbf{V}'}) \\ \downarrow {}^i\chi & & \downarrow \prod_{n \in \mathbb{Z}_{\geq 1}} {}^i\chi^n \\ {}^i\mathcal{F}_{\mathbf{V}} & \xrightarrow{\omega_i} & \prod_{n \in \mathbb{Z}_{\geq 1}} {}^i\mathcal{F}_{\mathbf{V}'}^n \end{array}$$

Hence,

$$\left(\prod_{n \in \mathbb{Z}_{\geq 1}} {}^i\chi^n \right) \circ \tilde{\omega}_i({}^i\mathcal{Q}_{\mathbf{V}}) \subset \omega_i \circ {}^i\chi({}^i\mathcal{Q}_{\mathbf{V}}).$$

Since $\omega_i({}^i\mathcal{F}_{\mathbf{V}}) \subset {}^i\mathcal{F}_{\mathbf{V}'}$,

$$\left(\prod_{n \in \mathbb{Z}_{\geq 1}} {}^i\chi^n \right) \circ \tilde{\omega}_i({}^i\mathcal{Q}_{\mathbf{V}}) \subset {}^i\mathcal{F}_{\mathbf{V}'}$$

For any two semisimple perverse sheaves \mathcal{L} and \mathcal{L}' in $\mathcal{D}_{G_{\mathbf{V}'}}({}^iE_{\mathbf{V}'})$ such that

$$\left(\prod_{n \in \mathbb{Z}_{\geq 1}} {}^i\chi^n \right)(\mathcal{L}) = \left(\prod_{n \in \mathbb{Z}_{\geq 1}} {}^i\chi^n \right)(\mathcal{L}'),$$

\mathcal{L} is isomorphic to \mathcal{L}' by Theorem III.12.1(3) in [7]. Since $(\prod_{n \in \mathbb{Z}_{\geq 1}} {}^i\chi^n)({}^i\mathcal{Q}_{\mathbf{V}'}) = {}^i\mathcal{F}_{\mathbf{V}'}$ and the objects in $\tilde{\omega}_i({}^i\mathcal{Q}_{\mathbf{V}})$ are semisimple, $\tilde{\omega}_i({}^i\mathcal{Q}_{\mathbf{V}}) \subset {}^i\mathcal{Q}_{\mathbf{V}'}$. □

Proposition 4.11. *We have the following commutative diagram*

$$\begin{array}{ccc} K({}_i\mathcal{Q}) & \xrightarrow{\tilde{\omega}_i} & K({}^i\mathcal{Q}) \\ \downarrow {}_i\chi & & \downarrow {}^i\chi \\ {}_i\mathcal{F}_{\mathcal{A}} & \xrightarrow{\omega_i} & {}^i\mathcal{F}_{\mathcal{A}} \end{array}$$

Proof. By the properties of ${}_i\chi^n$ and ${}^i\chi^n$, we have the following commutative diagram

$$\begin{array}{ccc} {}_i\mathcal{Q}_{\mathbf{V}} & \xrightarrow{\tilde{\omega}_i} & {}^i\mathcal{Q}_{\mathbf{V}'} \\ \downarrow {}_i\chi^n & & \downarrow {}^i\chi^n \\ {}_i\mathcal{F}_{\mathbf{V}}^n & \xrightarrow{\omega_i^n} & {}^i\mathcal{F}_{\mathbf{V}'}^n \end{array}$$

Hence, we get the commutative diagram in this proposition. \square

At last, Theorem 4.10 and Proposition 4.11 imply Theorem 3.5.

5. PROJECTIVE RESOLUTIONS OF A KIND OF STANDARD MODULES

5.1. KLR algebras. First let us review the definitions of KLR algebras ([6, 23]).

Let $Q = (I, H, s, t)$ be a quiver corresponding to the Lusztig's algebra \mathbf{f} . Let \mathbb{K} be an algebraic closed field. Fix an I -graded \mathbb{K} -vector space \mathbf{V} such that $\underline{\dim} \mathbf{V} = \nu \in \mathbb{N}I$. In Section 3.1, the semisimple perverse sheaves $\mathcal{L}_{\mathbf{i}} \in \mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})$ are defined for all $\mathbf{i} \in I^\nu$. Let

$$\mathcal{L}_\nu = \bigoplus_{\mathbf{i} \in I^\nu} \mathcal{L}_{\mathbf{i}}.$$

The KLR algebra \mathbf{R}_ν is defined as

$$\mathbf{R}_\nu = \bigoplus_{k \in \mathbb{Z}} \text{Ext}_{G_{\mathbf{V}}}^k(\mathcal{L}_\nu, \mathcal{L}_\nu).$$

\mathbf{R}_ν is a graded algebra and the degree of any element in $\text{Ext}_{G_{\mathbf{V}}}^k(\mathcal{L}_\nu, \mathcal{L}_\nu)$ is k .

Let $\mathbf{R}_\nu\text{-gmod}$ be the category of graded \mathbf{R}_ν -modules and $\mathbf{R}_\nu\text{-proj}$ be the category of finitely generated graded projective \mathbf{R}_ν -modules. Let $K(\mathbf{R}_\nu\text{-proj})$ be the Grothendieck group of $\mathbf{R}_\nu\text{-proj}$.

Define $v^\pm[P] = [P[\pm 1]]$. So $K(\mathbf{R}_\nu\text{-proj})$ is a free \mathcal{A} -module. Define

$$K(\mathbf{R}\text{-proj}) = \bigoplus_{\nu \in \mathbb{N}I} K(\mathbf{R}_\nu\text{-proj}).$$

For $\nu, \nu', \nu'' \in \mathbb{N}I$ such that $\nu = \nu' + \nu''$ and three I -graded \mathbb{K} -vector spaces $\mathbf{V}, \mathbf{V}', \mathbf{V}''$ such that $\underline{\dim} \mathbf{V} = \nu, \underline{\dim} \mathbf{V}' = \nu', \underline{\dim} \mathbf{V}'' = \nu''$, Khovanov and Lauda ([6]) defined a functor

$$\text{Ind}_{\nu', \nu''} : \mathbf{R}_{\nu'}\text{-proj} \times \mathbf{R}_{\nu''}\text{-proj} \rightarrow \mathbf{R}_\nu\text{-proj},$$

which induces an \mathcal{A} -bilinear multiplication

$$[\mathrm{Ind}_{\nu', \nu''}] : K(\mathbf{R}_{\nu'}\text{-proj}) \otimes_{\mathcal{A}} K(\mathbf{R}_{\nu''}\text{-proj}) \rightarrow K(\mathbf{R}_{\nu}\text{-proj}).$$

Khovanov and Lauda ([6]) proved that $K(\mathbf{R}\text{-proj})$ becomes an associative \mathcal{A} -algebra.

For any $\mathbf{y} \in Y_{\nu}$, let

$$P_{\mathbf{y}} = \bigoplus_{k \in \mathbb{Z}} \mathrm{Ext}_{G_{\mathbf{V}}}^k(\mathcal{L}_{\mathbf{y}}, \mathcal{L}_{\nu}).$$

Theorem 5.1 ([6, 20]). *There is a unique isomorphism of \mathcal{A} -algebras*

$$\gamma_{\mathcal{A}} : \mathbf{f}_{\mathcal{A}} \rightarrow K(\mathbf{R}\text{-proj})$$

such that $\gamma_{\mathcal{A}}(\theta_{\mathbf{y}}) = P_{\mathbf{y}}$ for all $\mathbf{y} \in Y_{\nu}$.

Let $\mathbf{B}_{\mathbb{Z}} = \{v^s b \mid b \in \mathbf{B}, s \in \mathbb{Z}\}$, which is a \mathbb{Z} -basis of $\mathbf{f}_{\mathcal{A}}$. Varagnolo, Vasserot and Rouquier proved the following theorem.

Theorem 5.2 ([23, 21]). *The map $\gamma_{\mathcal{A}}$ takes $\mathbf{B}_{\mathbb{Z}}$ to the \mathbb{Z} -basis of $K(\mathbf{R}\text{-proj})$ consisting of all indecomposable projective modules.*

5.2. Projective resolutions. Let i and j be two vertices of the quiver Q such that there are no arrows from i to j . Let $N = \#\{j \rightarrow i\}$ and m be a non-negative integer such that $m \leq N$. Let $\nu^{(m)} = mi + j \in \mathrm{NI}$. Fix an I -graded \mathbb{K} -vector space $\mathbf{V}^{(m)}$ such that $\underline{\dim} \mathbf{V}^{(m)} = \nu^{(m)}$.

Denote by $\mathbf{1}_{iE_{\mathbf{V}^{(m)}}} \in \mathcal{D}_{G_{\mathbf{V}^{(m)}}}(iE_{\mathbf{V}^{(m)}})$ the constant sheaf on $iE_{\mathbf{V}^{(m)}}$. The following functor is defined in Section 3.2:

$$j_{\mathbf{V}^{(m)}}! : \mathcal{D}_{G_{\mathbf{V}^{(m)}}}(iE_{\mathbf{V}^{(m)}}) \rightarrow \mathcal{D}_{G_{\mathbf{V}^{(m)}}}(E_{\mathbf{V}^{(m)}}).$$

Define

$$\mathcal{E}^{(m)} = j_{\mathbf{V}^{(m)}}!(v^{-mN} \mathbf{1}_{iE_{\mathbf{V}^{(m)}}}) \in \mathcal{D}_{G_{\mathbf{V}^{(m)}}}(E_{\mathbf{V}^{(m)}})$$

and

$$K_m = \bigoplus_{k \in \mathbb{Z}} \mathrm{Ext}_{G_{\mathbf{V}^{(m)}}}^k(\mathcal{E}^{(m)}, \mathcal{L}_{\nu^{(m)}}).$$

K_m is an object in $\mathbf{R}_{\nu^{(m)}}\text{-gmod}$ for any m . Note that K_m is a standard module in the sense of Kato ([5]). We shall give projective resolutions of these standard modules. For convenience, the complex $j_{\mathbf{V}^{(m)}}!(\mathbf{1}_{iE_{\mathbf{V}^{(m)}}}) \in \mathcal{D}_{G_{\mathbf{V}^{(m)}}}(E_{\mathbf{V}^{(m)}})$ is also denoted by $\mathbf{1}_{iE_{\mathbf{V}^{(m)}}}$.

For each $m \geq p \in \mathbb{N}$, consider the following variety

$$\tilde{S}_p^{(m)} = \{(x, W) \mid x \in E_{\mathbf{V}^{(m)}}, W \subset V_i, \dim(W) = p, \mathrm{Im} \bigoplus_{h \in H, t(h)=i} x_h \subset W\}.$$

Let $\pi_p : \tilde{S}_p^{(m)} \rightarrow E_{\mathbf{V}^{(m)}}$ be the projection taking (x, W) to x and $S_p^{(m)} = \mathrm{Im} \pi_p$.

By the definitions of $S_p^{(m)}$, we have

$$E_{\mathbf{V}^{(m)}} = S_m^{(m)} \supset S_{m-1}^{(m)} \supset S_{m-2}^{(m)} \supset \cdots \supset S_0^{(m)}.$$

For each $1 \leq p \leq m$, let

$$\mathcal{N}_p^{(m)} = S_p^{(m)} \setminus S_{p-1}^{(m)}.$$

Denote by $i_p^{(m)} : S_{p-1}^{(m)} \rightarrow S_p^{(m)}$ the close embedding and $j_p^{(m)} : \mathcal{N}_p^{(m)} \rightarrow S_p^{(m)}$ the open embedding.

Define

$$I_p^{(m)} = (\pi_p)_! (\mathbf{1}_{\tilde{S}_p^{(m)}}) [\dim \tilde{S}_p^{(m)}].$$

In [13], Lusztig proved that $I_p^{(m)}$ are semisimple perverse sheaves in $\mathcal{D}_{G_{\mathbf{V}^{(m)}}}(E_{\mathbf{V}^{(m)}})$. Hence $I_p^{(m)}$ correspond to projective modules in \mathbf{R}_{ν} -proj.

The following theorem is the main result in this section.

Theorem 5.3. *For $\mathcal{E}^{(m)}$, there exists $s_m \in \mathbb{N}$. For each $s_m \geq p \in \mathbb{N}$, there exists $\mathcal{E}_p^{(m)} \in \mathcal{D}_{G_{\mathbf{V}^{(m)}}}(E_{\mathbf{V}^{(m)}})$ such that*

- (1) $\mathcal{E}_{s_m}^{(m)} = \mathcal{E}^{(m)}$ and $\mathcal{E}_0^{(m)}$ is the direct sum of some semisimple perverse sheaves of the form $I_{p'}^{(m)}[l]$;
- (2) for each $p \geq 1$, there exists a distinguished triangle

$$\mathcal{E}_p^{(m)} \longrightarrow \mathcal{G}_p^{(m)} \longrightarrow \mathcal{E}_{p-1}^{(m)} \longrightarrow,$$

where $\mathcal{G}_p^{(m)}$ is the direct sum of some semisimple perverse sheaves of the form $I_{p'}^{(m)}[l]$.

The proof of Theorem 5.3 will be given in Section 5.3.

Let

$$P_0^{(m)} = \bigoplus_{k \in \mathbb{Z}} \text{Ext}_{G_{\mathbf{V}^{(m)}}}^k(\mathcal{E}_0^{(m)}, \mathcal{L}_{\nu^{(m)}})$$

and

$$P_s^{(m)} = \bigoplus_{k \in \mathbb{Z}} \text{Ext}_{G_{\mathbf{V}^{(m)}}}^k(\mathcal{G}_p^{(m)}, \mathcal{L}_{\nu^{(m)}}) \quad (1 \leq s \leq m),$$

which are projective modules in $\mathbf{R}_{\nu^{(m)}}\text{-proj}$.

As a corollary of Theorem 5.3, we have the following theorem.

Theorem 5.4. *For any $N \geq m \in \mathbb{N}$, there exists a finite length projective resolution of K_m :*

$$0 \longrightarrow P_0^{(m)} \longrightarrow P_1^{(m)} \longrightarrow \cdots \longrightarrow P_{s_m-1}^{(m)} \longrightarrow P_{s_m}^{(m)} \longrightarrow K_m \longrightarrow 0.$$

□

In the case of finite type, Kato proved that the projective dimension of any standard module is finite ([4, 5]). Theorem 5.4 show that the projective dimensions of a kind of standard modules are also finite in the general case.

5.3. The proof of Theorem 5.3. For convenience, a sheaf $\mathcal{A} \in \mathcal{D}_{G_{\mathbf{V}(m)}}(E_{\mathbf{V}(m)})$ is called with Property **A**(m), if \mathcal{A} satisfies the following conditions. There exists $s_{\mathcal{A}} \in \mathbb{N}$. For each $s_{\mathcal{A}} \geq p \in \mathbb{N}$, there exists $\mathcal{A}_p \in \mathcal{D}_{G_{\mathbf{V}(m)}}(E_{\mathbf{V}(m)})$ such that

- (1) $\mathcal{A}_{s_{\mathcal{A}}} = \mathcal{A}$ and \mathcal{A}_0 is the direct sum of some semisimple perverse sheaves of the form $I_{p'}^{(m)}[l]$;
- (2) for each $p \geq 1$, there exists a distinguished triangle

$$\mathcal{A}_p \longrightarrow \mathcal{G}_p^{\mathcal{A}} \longrightarrow \mathcal{A}_{p-1} \longrightarrow ,$$

where $\mathcal{G}_p^{\mathcal{A}}$ is the direct sum of some semisimple perverse sheaves of the form $I_{p'}^{(m)}[l]$.

Theorem 5.3 means that $\mathcal{E}^{(m)}$ is with Property **A**(m).

For the proof of Theorem 5.3, we need the following lemma.

Lemma 5.5. *Fix any distinguished triangle*

$$\mathcal{A} \longrightarrow \mathcal{A}' \longrightarrow \mathcal{A}'' \longrightarrow ,$$

where $\mathcal{A}, \mathcal{A}', \mathcal{A}'' \in \mathcal{D}_{G_{\mathbf{V}(m)}}(E_{\mathbf{V}(m)})$. If \mathcal{A} and \mathcal{A}'' are with Property **A**(m), \mathcal{A}' is with Property **A**(m) and $s_{\mathcal{A}'} = s_{\mathcal{A}} + s_{\mathcal{A}''} + 1$.

Proof. We shall prove this lemma by induction on $s_{\mathcal{A}''}$.

- (1) For $s_{\mathcal{A}''} = 0$, \mathcal{A}'' is the direct sum of some semisimple perverse sheaves of the form $I_{p'}^{(m)}[l]$. Let $\mathcal{A}'_{s_{\mathcal{A}'}} = \mathcal{A}'$ and $\mathcal{A}'_p = \mathcal{A}_p[1]$ for any $0 \leq p \leq s_{\mathcal{A}} = s_{\mathcal{A}'} - 1$. Let $\mathcal{G}_{s_{\mathcal{A}'}}^{\mathcal{A}'} = \mathcal{A}''$ and $\mathcal{G}_p^{\mathcal{A}'} = \mathcal{G}_p^{\mathcal{A}}[1]$ for any $1 \leq p \leq s_{\mathcal{A}} = s_{\mathcal{A}'} - 1$. The distinguished triangle

$$\mathcal{A} \longrightarrow \mathcal{A}' \longrightarrow \mathcal{A}'' \longrightarrow$$

implies

$$\mathcal{A}'_{s_{\mathcal{A}'}} \longrightarrow \mathcal{G}_{s_{\mathcal{A}'}}^{\mathcal{A}'} \longrightarrow \mathcal{A}'_{s_{\mathcal{A}'}-1} \longrightarrow$$

and the distinguished triangles

$$\mathcal{A}_p \longrightarrow \mathcal{G}_p^{\mathcal{A}} \longrightarrow \mathcal{A}_{p-1} \longrightarrow$$

imply

$$\mathcal{A}'_p \longrightarrow \mathcal{G}_p^{\mathcal{A}} \longrightarrow \mathcal{A}'_{p-1} \longrightarrow$$

for $1 \leq p \leq s_{\mathcal{A}'} - 1$. Hence, \mathcal{A}' is with Property **A**(m).

- (2) Assume that the lemma is true for $s_{\mathcal{A}''} < k$, we shall prove the lemma for $s_{\mathcal{A}''} = k$.

Now we have the following two distinguished triangles

$$\mathcal{A}' \xrightarrow{u} \mathcal{A}'' \longrightarrow \mathcal{A}[1] \longrightarrow$$

and

$$\mathcal{A}'' \xrightarrow{v} \mathcal{G}_k^{\mathcal{A}''} \longrightarrow \mathcal{A}''_{k-1} \longrightarrow .$$

Then we can construct the following distinguished triangle

$$\mathcal{A}' \xrightarrow{vu} \mathcal{G}_k^{\mathcal{A}''} \longrightarrow \mathcal{B} \longrightarrow .$$

By the octahedral axiom, there exist two maps $f : \mathcal{A}[1] \rightarrow \mathcal{B}$ and $g : \mathcal{B} \rightarrow \mathcal{A}''_{k-1}$ such that the following diagram commutes and the third row is a distinguished triangle

$$\begin{array}{ccccccc} \mathcal{A}' & \xrightarrow{\text{id}} & \mathcal{A}' & & & & \\ \downarrow u & & \downarrow vu & & & & \\ \mathcal{A}'' & \xrightarrow{v} & \mathcal{G}_k^{\mathcal{A}''} & \longrightarrow & \mathcal{A}''_{k-1} & \longrightarrow & \\ \downarrow & & \downarrow & & \downarrow \text{id} & & \\ \mathcal{A}[1] & \xrightarrow{f} & \mathcal{B} & \xrightarrow{g} & \mathcal{A}''_{k-1} & \longrightarrow & \\ \downarrow & & \downarrow & & & & \end{array}$$

Consider the following distinguished triangle

$$\mathcal{A}[1] \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{A}''_{k-1} \longrightarrow .$$

Since $\mathcal{A}''_k = \mathcal{A}''$ is with Property **A**(m), \mathcal{A}''_{k-1} is also with Property **A**(m) and $s_{\mathcal{A}''_{k-1}} = k-1$. By the induction hypothesis, \mathcal{B} is with Property **A**(m) and $s_{\mathcal{B}} = s_{\mathcal{A}} + k$. Hence, for each $s_{\mathcal{B}} \geq p \in \mathbb{N}$, there exists $\mathcal{B}_p \in \mathcal{D}_{G_{\mathbf{V}(m)}}(E_{\mathbf{V}(m)})$ such that

- 1) $\mathcal{B}_{s_{\mathcal{B}}} = \mathcal{B}$ and \mathcal{B}_0 is the direct sum of some semisimple perverse sheaves of the form $I_{p'}^{(m)}[l]$;
- 2) for each $p \geq 1$, there exists a distinguished triangle

$$\mathcal{B}_p \longrightarrow \mathcal{G}_p^{\mathcal{B}} \longrightarrow \mathcal{B}_{p-1} \longrightarrow ,$$

where $\mathcal{G}_p^{\mathcal{B}}$ is the direct sum of some semisimple perverse sheaves of the form $I_{p'}^{(m)}[l]$.

Note that $s_{\mathcal{A}'} = s_{\mathcal{B}} + 1$. Let $\mathcal{A}'_{s_{\mathcal{A}'}} = \mathcal{A}'$ and $\mathcal{A}'_p = \mathcal{B}_p$ for any $0 \leq p \leq s_{\mathcal{B}} = s_{\mathcal{A}'} - 1$. Let $\mathcal{G}_{s_{\mathcal{A}'}}^{\mathcal{A}'} = \mathcal{G}_k^{\mathcal{A}''}$ and $\mathcal{G}_p^{\mathcal{A}'} = \mathcal{G}_p^{\mathcal{B}}$ for any $1 \leq p \leq s_{\mathcal{B}} = s_{\mathcal{A}'} - 1$. The distinguished triangle

$$\mathcal{A}' \xrightarrow{vu} \mathcal{G}_k^{\mathcal{A}''} \longrightarrow \mathcal{B} \longrightarrow$$

implies

$$\mathcal{A}'_{s_{\mathcal{A}'}} \xrightarrow{vu} \mathcal{G}_{s_{\mathcal{A}'}}^{\mathcal{A}'} \longrightarrow \mathcal{A}'_{s_{\mathcal{A}'}-1} \longrightarrow$$

and the distinguished triangles

$$\mathcal{B}_p \longrightarrow \mathcal{G}_p^{\mathcal{B}} \longrightarrow \mathcal{B}_{p-1} \longrightarrow ,$$

imply

$$\mathcal{A}'_p \longrightarrow \mathcal{G}_p^{\mathcal{A}} \longrightarrow \mathcal{A}'_{p-1} \longrightarrow$$

for $1 \leq p \leq s_{\mathcal{A}'} - 1$. Hence, \mathcal{A}' is with Property **A**(m).

By induction, the proof is finished. \square

For the proof of Theorem 5.3, we also need the following proposition.

Proposition 5.6. *For each $m \geq p \in \mathbb{N}$, there exists $\mathcal{C}_p^{(m)} \in \mathcal{D}_{G_{\mathbf{V}(m)}}(E_{\mathbf{V}(m)})$ such that*

- (1) $\mathcal{C}_m^{(m)} = I_m^{(m)}$ and $\mathcal{C}_0^{(m)} = v^{-mN}(\mathcal{L}_{mi} \otimes \mathbf{1}_{E_{\mathbf{V}(0)}})$;
- (2) for each $p \geq 1$, there exists a distinguished triangle

$$v^{a_p^{(m)}}(\mathcal{L}_{(m-p)i} \otimes \mathbf{1}_{E_{\mathbf{V}(p)}}) \longrightarrow \mathcal{C}_p^{(m)} \longrightarrow \mathcal{C}_{p-1}^{(m)} \longrightarrow ,$$

where $a_p^{(m)} = p(m-p) - mN$.

Proof. We shall construct $\mathcal{C}_p^{(m)}$ for each p by induction.

(1) For $p = m$, let $\mathcal{C}_m^{(m)} = I_m^{(m)}$. It is clear that $I_m^{(m)} \simeq v^{-mN} \mathbf{1}_{E_{\mathbf{V}(m)}}$, that is $\mathcal{C}_m^{(m)} \simeq v^{a_m^{(m)}} \mathbf{1}_{E_{\mathbf{V}(m)}}$.

(2) For each $p < m$, we shall construct $\mathcal{C}_p^{(m)}$ and show that it satisfies the following conditions:

- 1) there exists a distinguished triangle

$$v^{a_{p+1}^{(m)}}(\mathcal{L}_{(m-p-1)i} \otimes \mathbf{1}_{E_{\mathbf{V}(p+1)}}) \longrightarrow \mathcal{C}_{p+1}^{(m)} \longrightarrow \mathcal{C}_p^{(m)} \longrightarrow ;$$

- 2) $\mathcal{C}_p^{(m)} = \mathcal{L}_{(m-p)i} \otimes \hat{\mathcal{C}}_p^{(m)}$, where $\hat{\mathcal{C}}_p^{(m)} \in \mathcal{D}_{G_{\mathbf{V}(p)}}(E_{\mathbf{V}(p)})$ and $\hat{\mathcal{C}}_p^{(m)} \simeq v^{a_p^{(m)}} \mathbf{1}_{E_{\mathbf{V}(p)}}$.

First, We construct $\mathcal{C}_p^{(m)}$ for $p = m-1$. There is a distinguished triangle

$$(10) \quad (j_m^{(m)})_!(j_m^{(m)})^*(\mathcal{C}_m^{(m)}) \longrightarrow \mathcal{C}_m^{(m)} \longrightarrow (i_m^{(m)})_*(i_m^{(m)})^*(\mathcal{C}_m^{(m)}) \longrightarrow .$$

Since $\mathcal{C}_m^{(m)} \simeq v^{a_m^{(m)}} \mathbf{1}_{E_{\mathbf{V}(m)}}$,

$$(j_m^{(m)})_!(j_m^{(m)})^*(\mathcal{C}_m^{(m)}) \simeq v^{a_m^{(m)}} \mathbf{1}_{E_{\mathbf{V}(m)}},$$

and

$$(i_m^{(m)})_*(i_m^{(m)})^*(\mathcal{C}_m^{(m)}) \simeq v^{a_m^{(m)}} \mathbf{1}_{S_{m-1}^{(m)}} .$$

Let $\mathcal{C}_{m-1}^{(m)} = (i_m^{(m)})_*(i_m^{(m)})^*(\mathcal{C}_m^{(m)})$. By (10), there exists a distinguished triangle

$$v^{a_m^{(m)}} \mathbf{1}_{E_{\mathbf{V}(m)}} \longrightarrow \mathcal{C}_m^{(m)} \longrightarrow \mathcal{C}_{m-1}^{(m)} \longrightarrow .$$

Since the support of $\mathcal{C}_{m-1}^{(m)}$ is in $S_{m-1}^{(m)}$, it can be wrote as

$$\mathcal{C}_{m-1}^{(m)} = \mathcal{L}_i \otimes \hat{\mathcal{C}}_{m-1}^{(m)},$$

where $\hat{\mathcal{C}}_{m-1}^{(m)} \in \mathcal{D}_{G_{\mathbf{V}(m-1)}}(E_{\mathbf{V}(m-1)})$. We have $\mathcal{C}_{m-1}^{(m)} \simeq v^{a_m^{(m)}} \mathbf{1}_{S_{m-1}^{(m)}} = v^{-mN} \mathbf{1}_{S_{m-1}^{(m)}}$. Hence

$$v^{-(m-1)} \hat{\mathcal{C}}_{m-1}^{(m)} \simeq v^{-mN} \mathbf{1}_{E_{\mathbf{V}(m-1)}},$$

that is,

$$\hat{\mathcal{C}}_{m-1}^{(m)} \simeq v^{-mN} v^{m-1} \mathbf{1}_{E_{\mathbf{V}(m-1)}} \simeq v^{a_{m-1}^{(m)}} \mathbf{1}_{E_{\mathbf{V}(m-1)}}.$$

Now, we have constructed $\mathcal{C}_{m-1}^{(m)}$ satisfying the following conditions:

1) there exists a distinguished triangle

$$v^{a_m^{(m)}} \mathbf{1}_{E_{\mathbf{V}(p)}} \longrightarrow \mathcal{C}_m^{(m)} \longrightarrow \mathcal{C}_{m-1}^{(m)} \longrightarrow ;$$

2) $\mathcal{C}_{m-1}^{(m)} = \mathcal{L}_i \otimes \hat{\mathcal{C}}_{m-1}^{(m)}$, where $\hat{\mathcal{C}}_{m-1}^{(m)} \in \mathcal{D}_{G_{\mathbf{V}(m-1)}}(E_{\mathbf{V}(m-1)})$ and $\hat{\mathcal{C}}_{m-1}^{(m)} \simeq v^{a_{m-1}^{(m)}} \mathbf{1}_{E_{\mathbf{V}(m-1)}}$.

(3) Assume that we have constructed $\mathcal{C}_p^{(m)}$ satisfying the following conditions:

1) there exists a distinguished triangle

$$v^{a_{p+1}^{(m)}} (\mathcal{L}_{(m-p-1)i} \otimes \mathbf{1}_{E_{\mathbf{V}(p+1)}}) \longrightarrow \mathcal{C}_{p+1}^{(m)} \longrightarrow \mathcal{C}_p^{(m)} \longrightarrow ;$$

2) $\mathcal{C}_p^{(m)} = \mathcal{L}_{(m-p)i} \otimes \hat{\mathcal{C}}_p^{(m)}$, where $\hat{\mathcal{C}}_p^{(m)} \in \mathcal{D}_{G_{\mathbf{V}(p)}}(E_{\mathbf{V}(p)})$ and $\hat{\mathcal{C}}_p^{(m)} \simeq v^{a_p^{(m)}} \mathbf{1}_{E_{\mathbf{V}(p)}}$.

We shall construct $\mathcal{C}_{p-1}^{(m)}$. First, there is a distinguished triangle

$$(j_p^{(p)})_! (j_p^{(p)})^* (\hat{\mathcal{C}}_p^{(m)}) \longrightarrow \hat{\mathcal{C}}_p^{(m)} \longrightarrow (i_p^{(p)})_* (i_p^{(p)})^* (\hat{\mathcal{C}}_p^{(m)}) \longrightarrow .$$

Hence, we have

$$\begin{aligned} & \mathcal{L}_{(m-p)i} \otimes (j_p^{(p)})_! (j_p^{(p)})^* (\hat{\mathcal{C}}_p^{(m)}) \longrightarrow \mathcal{L}_{(m-p)i} \otimes \hat{\mathcal{C}}_p^{(m)} \\ (11) \quad & \longrightarrow \mathcal{L}_{(m-p)i} \otimes (i_p^{(p)})_* (i_p^{(p)})^* (\hat{\mathcal{C}}_p^{(m)}) \longrightarrow . \end{aligned}$$

Since $\hat{\mathcal{C}}_p^{(m)} \simeq v^{a_p^{(m)}} \mathbf{1}_{E_{\mathbf{V}(p)}}$,

$$(j_p^{(p)})_! (j_p^{(p)})^* (\hat{\mathcal{C}}_p^{(m)}) \simeq v^{a_p^{(m)}} \mathbf{1}_{E_{\mathbf{V}(p)}},$$

and

$$(i_p^{(p)})_* (i_p^{(p)})^* (\hat{\mathcal{C}}_p^{(m)}) \simeq v^{a_p^{(m)}} \mathbf{1}_{S_{p-1}^{(p)}}.$$

Let $\mathcal{C}_{p-1}^{(m)} = \mathcal{L}_{(m-p)i} \otimes (i_p^{(p)})_* (i_p^{(p)})^* (\hat{\mathcal{C}}_p^{(m)})$. By (11), there exists a distinguished triangle

$$v^{a_p^{(m)}} (\mathcal{L}_{(m-p)i} \otimes \mathbf{1}_{E_{\mathbf{V}(p)}}) \longrightarrow \mathcal{C}_p^{(m)} \longrightarrow \mathcal{C}_{p-1}^{(m)} \longrightarrow .$$

Since the support of $\mathcal{C}_{p-1}^{(m)}$ is in $S_{p-1}^{(m)}$, it can be wrote as

$$\mathcal{C}_{p-1}^{(m)} = \mathcal{L}_{(m-p+1)i} \otimes \hat{\mathcal{C}}_{p-1}^{(m)},$$

where $\hat{\mathcal{C}}_{p-1}^{(m)} \in \mathcal{D}_{G_{\mathbf{V}(p-1)}}(E_{\mathbf{V}(p-1)})$. Since

$$(i_p^{(p)})_*(i_p^{(p)})^*(\hat{\mathcal{C}}_p^{(m)}) \simeq v^{a_p^{(m)}} \mathbf{1}_{S_{p-1}^{(p)}},$$

we have

$$\mathcal{C}_{p-1}^{(m)} = \mathcal{L}_{(m-p)i} \otimes (i_p^{(p)})_*(i_p^{(p)})^*(\hat{\mathcal{C}}_p^{(m)}) \simeq v^{-(m-p)p} v^{a_p^{(m)}} \mathbf{1}_{S_{p-1}^{(m)}} \simeq v^{-mN} \mathbf{1}_{S_{p-1}^{(m)}}.$$

Hence

$$v^{-(m-p+1)(p-1)} \hat{\mathcal{C}}_{p-1}^{(m)} = v^{-mN} \mathbf{1}_{E_{\mathbf{V}(p-1)}},$$

that is

$$\hat{\mathcal{C}}_{p-1}^{(m)} \simeq v^{-mN} v^{(m-p+1)(p-1)} \mathbf{1}_{E_{\mathbf{V}(p-1)}} \simeq v^{a_{p-1}^{(m)}} \mathbf{1}_{E_{\mathbf{V}(p-1)}}.$$

Now, we have constructed $\mathcal{C}_{p-1}^{(m)}$ satisfying the following conditions:

1) there exists a distinguished triangle

$$v^{a_p^{(m)}} (\mathcal{L}_{(m-p)i} \otimes \mathbf{1}_{E_{\mathbf{V}(p)}}) \longrightarrow \mathcal{C}_p^{(m)} \longrightarrow \mathcal{C}_{p-1}^{(m)} \longrightarrow ;$$

2) $\mathcal{C}_{p-1}^{(m)} = \mathcal{L}_{(m-p+1)i} \otimes \hat{\mathcal{C}}_{p-1}^{(m)}$, where $\hat{\mathcal{C}}_{p-1}^{(m)} \in \mathcal{D}_{G_{\mathbf{V}(p-1)}}(E_{\mathbf{V}(p-1)})$ and $\hat{\mathcal{C}}_{p-1}^{(m)} \simeq v^{a_{p-1}^{(m)}} \mathbf{1}_{E_{\mathbf{V}(p-1)}}$.

By induction, the proof is finished. \square

In Section 4.1, we have $\chi : K(\mathcal{Q}) \rightarrow \mathcal{F}$. In this section, we identify the Lusztig's algebra \mathbf{f} with the corresponding composition subalgebra \mathcal{F} .

Lusztig proved the following theorem.

Theorem 5.7 ([13]). $\chi(I_p^{(m)}) = \theta_i^{(m-p)} \theta_j \theta_i^{(p)}$ for each $m \geq p \in \mathbb{N}$.

By Proposition 5.6 and Theorem 5.7, we have the following corollary.

Corollary 5.8. *We have the following formula in \mathbf{f}*

$$\theta_j \theta_i^{(m)} = \sum_{p=0}^m v^{b_p^{(m)}} \theta_i^{(m-p)} \chi(\mathcal{E}^{(p)}),$$

where $b_p^{(m)} = (p - N)(m - p)$.

Proof. By Proposition 5.6 and Theorem 5.7, we have

$$\theta_j \theta_i^{(m)} = \sum_{p=0}^m v^{a_p^{(m)}} \theta_i^{(m-p)} \chi(\mathbf{1}_{E_{\mathbf{V}(p)}}) = \sum_{p=0}^m v^{a_p^{(m)}} v^{pN} \theta_i^{(m-p)} \chi(\mathcal{E}^{(p)}).$$

Since $a_p^{(m)} + pN = b_p^{(m)}$, we have

$$\theta_j \theta_i^{(m)} = \sum_{p=0}^m v^{b_p^{(m)}} \theta_i^{(m-p)} \chi(\mathcal{E}^{(p)}).$$

\square

We shall use Lemma 5.5 and Proposition 5.6 to prove Theorem 5.3 by induction.

Proof of Theorem 5.3. We shall prove this result by induction on m .

- (1) For $m = 0$, $\mathcal{E}^{(0)} = I_0^{(0)}$. It is clear that $\mathcal{E}^{(0)}$ is with Property **A**(0).
- (2) For $m = 1$, by Proposition 5.6, there exists a distinguished triangle

$$v^{-N} \mathbf{1}_{iE_{\mathbf{V}(1)}} \longrightarrow \mathcal{C}_1^{(1)} \longrightarrow \mathcal{C}_0^{(1)} \longrightarrow ,$$

where $\mathcal{C}_1^{(1)} = I_1^{(1)}$ and $\mathcal{C}_0^{(1)} = v^{-N}(\mathcal{L}_i \otimes \mathbf{1}_{iE_{\mathbf{V}(0)}})$. Since $\mathcal{E}^{(0)} = I_0^{(0)}$,

$$\mathcal{C}_0^{(1)} = v^{-N}(\mathcal{L}_i \otimes \mathbf{1}_{iE_{\mathbf{V}(0)}}) = \mathcal{L}_i \otimes \mathcal{E}^{(0)}$$

is the direct sum of some semisimple perverse sheaves of the form $I_{p'}^{(1)}[l]$. Hence, $\mathcal{E}^{(1)} = v^{-N} \mathbf{1}_{iE_{\mathbf{V}(1)}}$ is with Property **A**(1).

- (3) Assume the $\mathcal{E}^{(k)}$ is with Property **A**(k) for all $k < m$. Let us prove $\mathcal{E}^{(m)}$ is with Property **A**(m).

For any $k < m$, there exists $s_k \in \mathbb{N}$. For each $s_k \geq p \in \mathbb{N}$, there exists $\mathcal{E}_p^{(k)} \in \mathcal{D}_{G_{\mathbf{V}(k)}}(E_{\mathbf{V}(k)})$ such that

- 1) $\mathcal{E}_{s_k}^{(k)} = \mathcal{E}^{(k)}$ and $\mathcal{E}_0^{(k)}$ is the direct sum of some semisimple perverse sheaves of the form $I_{p'}^{(k)}[l]$;
- 2) for each $p \geq 1$, there exists a distinguished triangle

$$\mathcal{E}_p^{(k)} \longrightarrow \mathcal{G}_p^{(k)} \longrightarrow \mathcal{E}_{p-1}^{(k)} \longrightarrow ,$$

where $\mathcal{G}_p^{(k)}$ is the direct sum of some semisimple perverse sheaves of the form $I_{p'}^{(k)}[l]$.

Hence, we have the following distinguished triangle for each $p \geq 1$

$$\mathcal{L}_{(m-k)i} \otimes \mathcal{E}_p^{(k)} \longrightarrow \mathcal{L}_{(m-k)i} \otimes \mathcal{G}_p^{(k)} \longrightarrow \mathcal{L}_{(m-k)i} \otimes \mathcal{E}_{p-1}^{(k)} \longrightarrow .$$

Denote $\tilde{\mathcal{E}}_p^{(k)} = \mathcal{L}_{(m-k)i} \otimes \mathcal{E}_p^{(k)}$ and $\tilde{\mathcal{G}}_p^{(k)} = \mathcal{L}_{(m-k)i} \otimes \mathcal{G}_p^{(k)}$. Then, we have

$$\tilde{\mathcal{E}}_p^{(k)} \longrightarrow \tilde{\mathcal{G}}_p^{(k)} \longrightarrow \tilde{\mathcal{E}}_{p-1}^{(k)} \longrightarrow .$$

Because $\tilde{\mathcal{E}}_0^{(k)}$ and $\tilde{\mathcal{G}}_p^{(k)}$ are the direct sums of some semisimple perverse sheaves of the form $I_{p'}^{(m)}[l]$, $\tilde{\mathcal{E}}_k^{(k)}$ is with Property **A**(m). Since

$$\tilde{\mathcal{E}}_k^{(k)} = \mathcal{L}_{(m-k)i} \otimes \mathcal{E}_k^{(k)} = v^{-kN}(\mathcal{L}_{(m-k)i} \otimes \mathbf{1}_{iE_{\mathbf{V}(k)}}),$$

$\mathcal{L}_{(m-k)i} \otimes \mathbf{1}_{iE_{\mathbf{V}(k)}}$ is with Property **A**(m).

By Proposition 5.6, for each $m \geq k \in \mathbb{N}$, there exists $\mathcal{C}_k^{(m)} \in \mathcal{D}_{G_{\mathbf{V}(m)}}(E_{\mathbf{V}(m)})$ such that

- 1) $\mathcal{C}_m^{(m)} = I_m^{(m)}$ and $\mathcal{C}_0^{(m)} = v^{-mN}(\mathcal{L}_{mi} \otimes \mathbf{1}_{iE_{\mathbf{V}(0)}})$;

2) for each $k \geq 1$, there exists a distinguished triangle

$$v^{a_k^{(m)}}(\mathcal{L}_{(m-k)i} \otimes \mathbf{1}_{E_{\mathbf{V}(k)}}) \longrightarrow \mathcal{C}_k^{(m)} \longrightarrow \mathcal{C}_{k-1}^{(m)} \longrightarrow .$$

We have proved that $\mathcal{C}_0^{(m)}$ and $\mathcal{L}_{(m-k)i} \otimes \mathbf{1}_{E_{\mathbf{V}(k)}}$ ($1 \leq k \leq m-1$) are with Property **A**(m). Hence, by Lemma 5.5, $\mathcal{C}_{m-1}^{(m)}$ is with Property **A**(m). At last, by Lemma 5.5 and the distinguished triangle

$$\mathcal{C}_{m-1}^{(m)}[-1] \longrightarrow v^{-mN} \mathbf{1}_{E_{\mathbf{V}(p)}} \longrightarrow I_m^{(m)} \longrightarrow ,$$

$\mathcal{E}^{(m)} = v^{-mN} \mathbf{1}_{E_{\mathbf{V}(p)}}$ is with Property **A**(m).

By induction, the proof is finished. □

As a corollary of Theorem 5.3, we have

Corollary 5.9. *For each $N \geq m \in \mathbb{N}$, we have the following formula*

$$\chi(\mathcal{E}^{(m)}) = \sum_{p=0}^m (-1)^p v^{-p(1+N-m)} \theta_i^{(p)} \theta_j \theta_i^{(m-p)} = f(i, j; m).$$

Proof. By Theorem 5.3, we have

$$\chi(\mathcal{E}^{(m)}) = \sum_{p=0}^m c_p^{(m)} \theta_i^{(p)} \theta_j \theta_i^{(m-p)}.$$

We shall prove that $c_p^{(m)} = (-1)^p v^{-p(1+N-m)}$ ($0 \leq p \leq m$) by induction on m .

(1) For $m = 0$, by Corollary 5.8,

$$\theta_j = \chi(\mathcal{E}^{(0)}).$$

That is $c_0^{(0)} = 1$. Hence, the corollary is true in this case.

(2) Assume that $c_p^{(k)} = (-1)^p v^{-p(1+N-k)}$ ($0 \leq p \leq k$) for any $k < m$. We shall prove that $c_q^{(m)} = (-1)^q v^{-q(1+N-m)}$ ($0 \leq q \leq m$).

By Corollary 5.8,

$$\begin{aligned} \theta_j \theta_i^{(m)} &= \sum_{k=0}^m v^{b_k^{(m)}} \theta_i^{(m-k)} \chi(\mathcal{E}^{(k)}) \\ &= \sum_{k=0}^{m-1} v^{b_k^{(m)}} \theta_i^{(m-k)} \chi(\mathcal{E}^{(k)}) + v^{b_m^{(m)}} \chi(\mathcal{E}^{(m)}) \\ &= \sum_{k=0}^{m-1} v^{b_k^{(m)}} \theta_i^{(m-k)} \chi(\mathcal{E}^{(k)}) + \chi(\mathcal{E}^{(m)}). \end{aligned}$$

Hence,

$$\begin{aligned}
\chi(\mathcal{E}^{(m)}) &= \theta_j \theta_i^{(m)} - \sum_{k=0}^{m-1} v^{b_k^{(m)}} \theta_i^{(m-k)} \chi(\mathcal{E}^{(k)}) \\
&= \theta_j \theta_i^{(m)} - \sum_{k=0}^{m-1} v^{b_k^{(m)}} \theta_i^{(m-k)} \sum_{p=0}^k c_p^{(k)} \theta_i^{(p)} \theta_j \theta_i^{(k-p)} \\
&= \theta_j \theta_i^{(m)} - \sum_{k=0}^{m-1} \sum_{p=0}^k v^{b_k^{(m)}} c_p^{(k)} \frac{[m-k+p]_v!}{[m-k]_v! [p]_v!} \theta_i^{(m-k+p)} \theta_j \theta_i^{(k-p)}.
\end{aligned}$$

For any $q \geq 1$,

$$\begin{aligned}
c_q^{(m)} &= - \sum_{k=m-q}^{m-1} v^{b_k^{(m)}} c_{q+k-m}^{(k)} \frac{[q]_v!}{[m-k]_v! [q+k-m]_v!} \\
&= - \sum_{k=0}^{q-1} v^{b_{k+m-q}^{(m)}} c_k^{(k+m-q)} \frac{[q]_v!}{[q-k]_v! [k]_v!}.
\end{aligned}$$

By the induction hypothesis,

$$\begin{aligned}
c_q^{(m)} &= - \sum_{k=0}^{q-1} v^{b_{k+m-q}^{(m)}} c_k^{(k+m-q)} \frac{[q]_v!}{[q-k]_v! [k]_v!} \\
&= - \sum_{k=0}^{q-1} (-1)^k v^{(k+m-q-N)(q-k)} v^{-k(1+N-k-m+q)} \frac{[q]_v!}{[q-k]_v! [k]_v!} \\
&= -v^{q(m-q-N)} \sum_{k=0}^{q-1} (-1)^k v^{k(q-1)} \frac{[q]_v!}{[q-k]_v! [k]_v!} \\
&= -v^{q(m-q-N)} \sum_{k=0}^q (-1)^k v^{k(q-1)} \frac{[q]_v!}{[q-k]_v! [k]_v!} + v^{q(m-q-N)} (-1)^q v^{q(q-1)} \\
&= v^{q(m-q-N)} (-1)^q v^{q(q-1)} = (-1)^q v^{-q(1+N-m)}
\end{aligned}$$

Note that

$$c_0^{(m)} = 1 = (-1)^0 v^{-0(1+N-m)}.$$

Hence, $c_q^{(m)} = (-1)^q v^{-q(1+N-m)}$ for any $0 \leq q \leq m$.

By induction, for each $N \geq m \in \mathbb{N}$, $c_p^{(m)} = (-1)^p v^{-p(1+N-m)}$ ($0 \leq p \leq m$) and

$$\chi(\mathcal{E}^{(m)}) = \sum_{p=0}^m (-1)^p v^{-p(1+N-m)} \theta_i^{(p)} \theta_j \theta_i^{(m-p)}.$$

□

5.4. The formulas of Lusztig's symmetries. In this section, we shall give a new proof of Proposition 2.1.

Consider the following quiver

$$Q : i \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} j$$

with vertex set $I = \{i, j\}$ and N arrows from j to i . Let $Q' = \sigma_i Q$ be the quiver by reversing the directions of all arrows

$$Q' : i \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} j$$

Let m be a non-negative integer such that $m \leq N$ and $m' = N - m$. Let $\nu = mi + j \in \mathbb{N}I$ and $\nu' = s_i \nu = m'i + j \in \mathbb{N}I$. Fix two I -graded \mathbb{K} -vector spaces \mathbf{V} and \mathbf{V}' such that $\underline{\dim} \mathbf{V} = \nu$ and $\underline{\dim} \mathbf{V}' = \nu'$.

Denote by $\mathbf{1}_{iE_{\mathbf{V},Q}} \in \mathcal{D}_{G_{\mathbf{V}}}(iE_{\mathbf{V},Q})$ the constant sheaf on $iE_{\mathbf{V},Q}$ and $\mathbf{1}_{iE_{\mathbf{V}',Q'}} \in \mathcal{D}_{G_{\mathbf{V}'}}(iE_{\mathbf{V}',Q'})$ the constant sheaf on $iE_{\mathbf{V}',Q'}$. For convenience, denote $iE_{\mathbf{V},Q}$ (resp. $iE_{\mathbf{V}',Q'}$) by $iE_{\mathbf{V}}$ (resp. $iE_{\mathbf{V}'}$) and $\mathbf{1}_{iE_{\mathbf{V},Q}}$ (resp. $\mathbf{1}_{iE_{\mathbf{V}',Q'}}$) by $\mathbf{1}_{iE_{\mathbf{V}}}$ (resp. $\mathbf{1}_{iE_{\mathbf{V}'}}$).

Denote

$$\mathcal{E}^{(m)} = j_{\mathbf{V}!}(v^{-mN} \mathbf{1}_{iE_{\mathbf{V}}}) \in \mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})$$

and

$$\mathcal{E}'^{(m')} = j_{\mathbf{V}'!}(v^{-m'N} \mathbf{1}_{iE_{\mathbf{V}'}}) \in \mathcal{D}_{G_{\mathbf{V}'}}(E_{\mathbf{V}'}).$$

In Section 3.3, we give the following geometric realization of the Lusztig's symmetry T_i :

$$\tilde{\omega}_i : \mathcal{D}_{G_{\mathbf{V}}}(iE_{\mathbf{V}}) \rightarrow \mathcal{D}_{G_{\mathbf{V}'}}(iE_{\mathbf{V}'}).$$

Proposition 5.10. *For any $N \geq m \in \mathbb{N}$, $\tilde{\omega}_i(v^{-mN} \mathbf{1}_{iE_{\mathbf{V}}}) = v^{-m'N} \mathbf{1}_{iE_{\mathbf{V}'}}$.*

Proof. By the definitions of α and β in the diagram (5) of Section 3.3,

$$\alpha^*(\mathbf{1}_{iE_{\mathbf{V}}}) = \mathbf{1}_{Z_{\mathbf{V}\mathbf{V}'}} = \beta^*(\mathbf{1}_{iE_{\mathbf{V}'}}).$$

Hence

$$\tilde{\omega}_i(\mathbf{1}_{iE_{\mathbf{V}}}) = v^{(m-m')N} \mathbf{1}_{iE_{\mathbf{V}'}}.$$

That is

$$\tilde{\omega}_i(v^{-mN} \mathbf{1}_{iE_{\mathbf{V}}}) = v^{-m'N} \mathbf{1}_{iE_{\mathbf{V}'}}.$$

□

Corollary 5.9 implies $\chi(\mathcal{E}^{(m)}) = f(i, j; m)$. Similarly, we have $\chi(\mathcal{E}'^{(m')}) = f'(i, j; m')$. Hence, Proposition 5.10 implies Proposition 2.1.

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